A NEW APPROACH TO AERIAL COMBAT GAMES

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By Sheldon Baron, Kai-Ching Chu,[*] Yu-Chi Ho,[*]
and David L. Kleinman

[*]Harvard University

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ABSTRACT

The salient results of a study to apply differential game theory to aerial combat problems are presented in this report. A new approach to manned aerial combat games has been developed that includes the statistical effects of human observation. This method has been applied to two classical differential game problems. In addition, an approach to the "dogfight" problem has been developed.

FOREWORD

This report was prepared by the Control Systems Department of Bolt Beranek and Newman Inc., Cambridge, Massachusetts. It represents the results of a study made for the Langley Research Center under NASA Contract NAS1-8296 that resulted from an unsolicited proposal. The work was administered under the direction of Dr. John D. Bird of the Theoretical Mechanics Branch, ASMD at Langley.

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1. INTRODUCTION

Until recently, aerial combat problems had been of decreasing interest to the research community. The evolving nature of warfare, increases in aircraft speeds, and advances in weapons systems all seemed to mitigate against manned aerial "duels". Hence, aircraft performance characteristics were most often being determined by other mission requirements and the pilot's role in any aerial combat appeared to be of lessening importance. Indeed, the "dogfight", where slight advantages in aircraft performance and pilot skill would determine the outcome in an aerial battle, seemed to be a thing of the past.

Recent developments in Southeast Asia have shown that this evaluation of air-to-air combat was a bit premature — the days of the "Red Baron" are far from over (Ref.1). Moreover, questions are being raised anew as to requisite aircraft performance characteristics and handling qualities, appropriate pilot tactics, procedures for pilot training, etc. As a result, research into aerial combat problems has been stimulated greatly.

In its simplest form, aerial combat may be considered a problem of pursuit and evasion. One aircraft, the pursuer, is attempting to "capture" an opposing aircraft, the evader, who wishes to avoid such an outcome.[†] (We use the word capture to designate an outcome that is advantageous to the pursuer. For example, capture may mean simply that the evader is within target of the pursuer's guns.) In a more complex version of the problem, the roles of pursuer and evader may be interchanged several times during the course of a battle; such a situation may be called,

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[†]For ease of discussion, we shall frequently refer to the aircraft as though they were opposing individuals.

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appropriately, a "dogfight". In either case, aerial combat, as the name implies, is a situation in which the participants have conflicting objectives. The mathematical theory which is concerned with determining optimal courses of action for opponents with conflicting objectives is the Theory of Games (Ref.2). That portion of the theory which deals with situations of a dynamic nature, involving lengthy sequences of logically connected decisions, is called Differential Game Theory (Ref.3). It is this aspect of game theory that is most pertinent to the study of aerial combat.

This report describes the results of a research program[†] to investigate the application of differential game theory to aerial combat problems. Simplified combat problems were to be analyzed in an attempt to determine fundamental relations between the outcome of a combat and such factors as aircraft performance characteristics, pilot tactics and information available. In addition, the value of differential game theory for studying complex aerial combat problems was to be assessed. Before we discuss what we have accomplished in connection with these objectives, let us describe some of the salient aspects of the problem.

What is a Differential Game? This simple question is not answered simply. For present purposes, however, it is useful to describe briefly the most basic type of problem, the so-called zero-sum, deterministic differential game. A crude definition of such a game may be given as follows:[††]

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[†]The program was conducted by BBN for the Langley Research Center under Contract No. NAS 1-8296.

[††]A more precise formulation of the problem can be found in Ref.4.

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Determine the pair (u*,v*) that provides a saddlepoint of

$$J(t_o,x_o;u,v) = g(x(T),T) + \int_{t_o}^{T} L(x,u,v,t)dt \qquad (1.1)$$

subject to the constraints

$$\dot{x} = f(x,u,v,t) \quad ; \quad x(t_o) = x_o \qquad (1.2)$$

$$u\epsilon U \quad , \quad v\epsilon V \qquad (1.3)$$

and

$$h(x(T),T) = 0 \qquad (1.4)$$

where, in the parlance of game theory, J is the payoff, x is the (vector) position or "state" of the game, u and v are called strategies, and are restricted to certain sets of admissable strategies, U and V, which depend, in general, on the specific problem to be solved, and a saddlepoint is the pair of optimal strategies u* and v* which satisfy

$$J(t_o,x_o;u^*,v) \leq J(t_o,x_o;u^*,v^*) \leq J(t_o,x_o;u,v^*) \qquad (1.5)$$

Equations (1.2) through (1.4) can be thought of as defining the rules of the game. The progress of play is governed by the n first-order differential equations (1.2) — hence, the name Differential Game. Play starts at time t_o in the state x_o and terminates at the first time t=T such that (1.4) is satisfied, i.e., Eq. (1.4) provides a stopping rule.

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The payoff J is a numerical measure for determining the out-
come of the game and for evaluating the merits of particular
strategies. Conflict is introduced, mathematically, by having
one player choose his strategy to maximize the payoff, while the
other player attempts to minimize the same payoff through his
selection of a strategy. The game is zero-sum because there is a
single payoff and one player's gain is the other player's loss.

The optimal strategies u* and v* are complete prescriptions
for play of the game in terms of the information available. This
type of differential game is assumed to be one of perfect infor-
mation: both players know how the game proceeds (Eqs. (1.1)
through (1.4)) and the state x(t) at time t. As a result,
strategies are normally defined by expressions of the form

$$u = k(x(t),t)\epsilon U$$

(1.6)

$$v = \overline{k}(x(t),t)\epsilon V$$

In other words, strategies are feedback, or closed-loop, control
laws. When the players do not have perfect information, one can-
not obtain solutions in terms of strategies of the form of
Eq. (1.6), and alternate procedures are dictated. We shall say
more about this later.

It is important to note that there is no assurance that the
game, as formulated above, will ever terminate. Indeed, in some
instances, the very essence of the problem can be termination of
the game. For example, if Eq. (1.4) represents the condition for
capture, one may simply be interested in whether capture is poss-
ible, i.e., whether the game terminates. Isaacs (Ref.3) calls

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such a problem a "game of kind". (In Chapter 5 we will see that the concept of a "game of-kind" is useful in studying the "dog-fight" problem.)

To study aerial combat in the context of the above game, one must specify the payoff (1.1), the equations of state (1.2), the control constraints (1.3), and the termination criteria (or constraints)(1.4). Ordinarily, the equations of state will be some set of equations describing the motion of the vehicles. Embodied in these equations and in the control constraints will be the performance capabilities of the two aircraft. The payoff and the termination criteria will generally reflect the goals of the combatants, but they might also include factors that are indicative of system constraints (e.g., weapons systems limitations). The assumptions that are made with respect to these various aspects of the problem will depend on the kind of results being sought. It is natural to expect, however, that the more realistic the assumptions, the harder it will be to solve the problem analytically. This is indeed true and anything short of linearizing the equations of state and making special assumptions about the payoff and the constraints leads to problems that are intractable analytically. Of course, if one attempts to further complicate the problem, say by including stochastic effects, the difficulties are compounded.

There are two obvious approaches that might be employed in attempting to overcome these difficulties. First, one can attempt to formulate meaningful simplified problems that can be solved by analytic, geometric or specialized computational procedures. The solutions to such problems could be most useful for obtaining general insights into the nature of aerial combat, provided that one appreciates the effects that the requisite simplifying assumptions have on the results. Second, one

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can attempt to develop general computational methods for air-combat "games" so that both simplified and realistic problems can be solved directly.

To some extent, we have pursued both approaches in this program. We began by analyzing simplified problems with the aims of exploring the difficulties associated with these problems and of developing the techniques and insights necessary to solve more realistic problems. These investigations included a preliminary study of the dogfight problem that revealed some of the essential characteristics of this exceedingly difficult problem. We also studied the Homocidal Chauffeur problem in some depth, examining the effects of: (i) changing the capture region to a fan-shaped "firing envelope"; (ii) using a penalty function in place of terminal constraints; (iii) constraining the evader's control; and, (iv) degrading the information available to the players (i.e., imperfect information). These latter investigations, though interesting and informative, turned out to be ancillary to the research effort and, therefore, they will not be discussed in this report. As a result of our analyses, we have obtained a better understanding of the nature of differential game solutions, and of various aspects of the air-combat problem, although we cannot assert that we have found the fundamental relations that we originally sought.

The analytical investigations demonstrated the need for a general computational approach to air-combat games. They showed that it is even difficult to solve problems that *appear* to be comparatively simple. Consequently, the major portion of our work was devoted to the search for a suitable computation scheme. Our studies have led us to a new conceptualization of the air-combat problem and a concomitant computational technique. The

approach will be described in detail later in this report. In essence, it involves taking account of pilot limitations directly and in such a way that the aerial-combat problem can be reduced to a Markov game. Although the method has not been fully tested, it appears that it has many advantages and that it has great potential for solving realistic aerial-combat problems. We believe that the discovery and preliminary development of this approach is the most significant result of the research performed in this program.

In the remainder of this report we shall discuss in detail the results of our investigations. This chapter and the next one are devoted primarily to background material and to an effort to place our work in proper perspective with other research on differential game theory and its application to air-combat problems. In Chapter 3, the above-mentioned computational approach to air-combat "games" is described in detail. Results obtained in applying the technique to two example problems are given in the next chapter. Our analysis of the dogfight problem receives separate attention in Chapter 5. Finally, in Chapter 6, we present some concluding remarks and suggestions for further research.

2. THE ROLE OF DIFFERENTIAL GAMES IN AERIAL COMBAT STUDIES

The theory of differential games has sometimes been heralded as the "answer" to the aerial combat problem (just as optimal control theory was the solution to the trajectory optimization problem). On the other hand, opinions have also been expressed that the theory will never be made practical enough to have a significant impact on such problems (Ref.5). In the long run, one may expect the truth to lie somewhere between these extreme points of view. But, what is the *current* role of differential game theory in aerial combat problems and what can be expected in the near future?

In an attempt to shed some light on these questions and to place in perspective the work to be described in later chapters, we will now examine the actual state-of-the-art with respect to differential games. However, we will not attempt to present an exhaustive review of the literature. Instead, we try to discuss issues that, once clarified, make it easier to assess the difficulties and potentialities of differential game theory. We also examine the application of differential game theory to aerial combat problems from the standpoint of what has been done, what people are doing, and what prospects are for the future.

State of the Theory

The historical origin of differential games (Ref.3) and its concurrent development with optimal control theory are well-known (Ref.6). When the connection between the two was realized belatedly, and popularized in the early- and mid-sixties, a flurry of activity followed and attempts were made to unify the two disciplines. Inasmuch as most of the active workers came from the

optimal control field, a natural tendency was to view differential games as an extension of optimal control theory. While some success is achieved with this viewpoint (Ref.7), it gradually became evident that such an approach is not entirely satisfactory. Differential game theory is so full of pitfalls, analytical and conceptual, that experts as well as the unsuspecting fall victim to them.

Given the difficulties associated with differential game problems, and the popular notion that differential game theory was merely an extension of optimal control theory, it is not surprising that much of the differential games literature is devoted to two-player, zero-sum, deterministic problems. While these problems are of undeniable importance, preoccupation with them leads to an unnecessarily narrow point of view. The role of differential games in air combat problems is more readily assessed if we first expand our outlook. This can be done by considering differential games (and, hence, optimal control problems) as a special case of a larger class of dynamic optimization problems. The framework for this larger class of problems may be called *Generalized Control Theory*.

Differential Games and Generalized Control Theory--Our prime concern here is with three aspects of dynamic optimization problems: the performance measures, the number of controllers, and the information available to these controllers. In traditional optimal control problems, there is a single performance measure and a single controller who is supposed to coordinate all control actions. This controller has access to all the available information, although the specific form and type of information may vary from problem to problem. In the zero-sum, deterministic game defined in Chapter 1, there are two controllers both of whom

have access to the same "perfect" information set. Each controller has his own performance measure, but, since one is the negative of the other (the zero-sum property), a single criterion may be used in the formulation of the game. Thus, with respect to the three aspects of an optimization problem being considered here, this type of game differs only slightly from the optimal control problem. (However, in terms of solving problems, the implications of this slight difference are substantial.)

Many important problems can be formulated either as optimal control problems or as two-player, zero-sum, deterministic differential games. On the other hand, it is easy to visualize realistic situations or problems that cannot be treated adequately within either of these frameworks. Indeed, it is not difficult to think of problems in which there is a multiplicity of controllers (operating with or without cooperation from others), each having a different information set and a different payoff (performance measure). We call such a multi-controller, multi-information-set, multi-payoff situation a *generalized control problem*.

The notion of generalized control problems is relatively new. As a start toward the development of a theory for such problems, it is useful to classify specific problems within the generalized framework. One way of doing this is to divide into subcategories each of the aspects of an optimization problem that we have been discussing. For the criterion J, we have: (i) one J, (ii) two J's with $J_1 = -J_2$, and (iii) multiple J's with $\Sigma J_i \neq 0$.[†] For the controller C, we have: (i) one C, (ii) two C's, and (iii) multiple C's. Finally, for the information set I, we have the cases: (i) one perfect information set, (ii) one noisy (or imperfect) information set, and (iii) multiple information sets.

[†] Zero-sum games with more than two players may also be defined, but they are of minor interest.

Using the above categories, one can organize various problem
areas in the manner shown in Table 1. From this table we can see
why the generalized control theory viewpoint is important concep-
tually.[†] For instance, it shows clearly that problems of conflict
are just a special class of the general optimization problem.
Furthermore, two-player, zero-sum, deterministic differential
games are, in turn, only a particular case of the problem of con-
flict. Thus, if the theory of such games is what one means by
differential game theory (as much of the literature would indicate),
it is indeed bold to claim that this theory will ultimately pro-
vide the solution to all aerial combat problems. Alternatively,
to assert that differential game theory will be of no utility for
aerial combat studies, one must not only be convinced of the
futility of the zero-sum approach, he must also be ignorant of
the much richer problem area implied by generalized control theory.

Given that generalized control theory is significant concep-
tually and that solutions to the more general problems posed by
it would be extremely useful, it is natural to ask:"What problems
are raised by adopting this broader outlook?" It seems to us that
the most important and most relevant problems are those associated
with performance measures and information structures. We now dis-
cuss briefly some of these problems.

While it has long been recognized that a single (scalar)
criterion is often not an adequate measure of system performance,
it has not been generally recognized that it is possible to con-
sider simultaneously several criterion functions. The chief
price one pays for such consideration is that it is not clear what
one means by a "solution" to a vector-valued optimization problem.

[†]The table indicates that the names for the various problem cate-
gories may be inadequate. For example, a more definitive title
for (3) is "a 2C/2J/PI problem in generalized control theory."
However, the names are firmly entrenched and one can hardly ex-
pect such a new terminology to take hold at this stage.

TABLE 1
Classification of Optimization Problems

	Controller			Criterion			Information Set			
	One	Two	Multiple	One	Two $J_1 = -J_2$	Multiple	One Perfect	One Imperfect	Multiple	References
(1) Deterministic Optimal Control	✓			✓			✓			
(2) Stochastic Optimal Control	✓			✓				✓		
(3) Zero-Sum Differential Game		✓			✓		✓			
(4) Stochastic Zero-Sum Differential Game		✓			✓				✓	
(5) Vector Value Optimization Problem (Negotiation pb.)	✓					✓	✓			
(6) Nonzero-Sum Differential Games			✓			✓	✓			
(7) Team Theory			✓	✓					✓	
(8) MC/MJ/MI pb.			✓			✓			✓	

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Many new concepts, such as the "prisoner's dilemna", the Nash equilibrium, the negotiation set, and the minimax solution, arise (Ref.8). However, this ambiguity merely forces one to sharpen his definition of optimality, actually leading to a better understanding of the multi-criterion problem. We realize, for example, that the often used device of optimizing one criterion that is a weighted sum of the many criteria that require consideration is but one of the many ways of resolving the problem (Refs.9,10). Methods for performing this kind of reduction are described in Refs. 11 and 12.

With respect to aerial combat, it is reasonably clear that a single criterion is not adequate for certain fighter vs. fighter duel situations. In a dogfight, for example, the combatants may change roles from pursuer to evader many times during the course of an engagement and the goals associated with the two roles are obviously different. In addition, pilot preferences and overall tactical considerations enter the problem in a nontrivial way. We shall have more to say about some of these matters in Chapter 5 where we discuss the dogfight problem.

One of the more interesting aspects of generalized control theory is the problem of information structures. As noted earlier, in traditional control problems the information pattern is always assumed to be complete in the sense that the controller has access to all the information available, whether it be perfect (deterministic) or imperfect (stochastic). This cherished concept is also carried over to the zero-sum differential game. However, even for these games we begin to get an inkling of the effect of information on the outcome. (Ref.13) We find that open-loop (no information) and closed-loop (perfect information) plays give rise to different conditions for the existence of saddlepoint solutions.

This has important implications with respect to the numerical "solutions" of differential games.

In more general cases, the effect of information patterns can be overriding. Team theory (Refs.14,15) is essentially a study of the value of information in decentralized optimization problems. Strategic missile defense and squadron vs. squadron air duels are basically team theoretic problems (e.g., what is the value of the information that lets every fighter or missile know the intended target of all fighters or missiles on its team? Or, in other words, how much should we pay for completely centralized control?). In nonzero-sum differential games, information patterns can have a profound, and sometimes surprising, effect. For example, it can be shown that in some cases open-loop control can actually outperform closed-loop control (to the benefit of both players) (Ref.16). Finally, in all cases where there is more than one controller and more than one imperfect information set, the fundamental problem of information closure arises. This is the familiar problem of "I know that you know that I know that you know.....". So far, this problem has only been resolved for the special class of zero-sum stochastic differential games (Refs.16-18), and open-loop team-theoretic problems (Ref.14), involving linear systems, quadratic criteria, and gaussian noise. In these cases, the optimal actions turn out to be linear functions of the information available to each player. This is not necessarily an expected result inasmuch as one can cite a counter example of a two-stage, closed-loop team-theory problem for which the linear controller is non-optimal (Ref.19).

Zero-Sum Deterministic Differential Games--We have just seen that zero-sum deterministic differential games[†] are a special

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[†]Hereafter, unless noted otherwise, we shall mean this type of game when we speak of differential games.

class of what we called generalized control problems. As such, these games are not subject to most of the difficulties concerning performance measures and information structures that were described in the previous section. However, even if we restrict ourselves to this class of problems, our troubles are far from over. From an analytic standpoint, the prime source of difficulty is that global differential game solutions are frequently characterized by the existence of a variety of singular surfaces (Refs.20-23). Consequently, techniques for obtaining answers that rely on "smoothness" assumptions must be used with great care; solutions cannot be propagated readily across the singular surfaces. To make matters worse, there do not appear to be generally effective procedures for the treatment of these surfaces. They seem to tax the ingenuity and resourcefulness of the best of researchers. However, from a practical point of view, many of the singular surfaces are somewhat less important. To a large extent, their existence depends on the infinite divisibility of time and space and, in practice, these quantities will often be discretized. Isaacs' book (Ref.3) contains the most complete discussion of these singular surfaces. They will not be pursued further here.

The numerical solution of differential games is very difficult. However, if one wishes to determine the optimal open-loop control against an opponent operating with known guidance logic, that may or may not be optimized, then the problem can be solved for reasonably complex situations.[†] Although these are useful kinds of results, it must be recognized that they do not represent true differential game solutions. They are simply solutions to an optimal control problem in which the control of one of the players has been predetermined. General numerical techniques for obtaining true differential game solution do not

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[†]For example, for problems involving, say, 15 state variables (Private Communication: B. Morgan, AF Academy).

exist. Of course, this state of affairs should not be surprising since we cannot even solve the general closed-loop optimal control problem. Suboptimal closed-loop solutions to differential games may be obtained using successive approximation procedures. The technique is essentially to optimize, repeatedly and alternately, the strategy of one player while holding fixed the strategy of the other player. It can be shown that the successive maximum and minimum values attained by the payoff bound the saddlepoint value.

Aerial Combat Problems

To our knowledge, the most complete survey of aerial combat problems is contained in the 1967 Rand report by Greene and Huntzicker (Ref.24). They divide research in this area into the six categories listed below.

(1) Instructional material, historical descriptions, and test documentation.

(2) Studies of subsystems: These studies deal mostly with performance of ordnance subsystems and of aircraft. (A notable example of the aircraft performance studies is the "energy maneuverability" work of Rutowski (Ref.25).)

(3) Game-theoretic studies of duels: Despite the title, this work does not include research in differential games. Instead, it covers early efforts to apply ordinary game theory to problems of ordnance selection and timing of firing.

(4) Detailed studies of combined system performance (fighter vs. predictable target): Fighter vs. bomber problems are typical of the examples studied here. These efforts were inspired largely by post World War II interest in bomber defense.

(5) Air-Battle simulations: The problems here involve extensive simulation studies of aerial battles involving squadrons of aircraft. Resource allocation, rather than the details of control action, is the primary item of interest in these studies.

(6) Duel between maneuvering vehicles: As noted earlier, this type of problem has become important again as a result of Vietnam.

We shall devote the rest of this discussion to the last of the above categories, inasmuch as this study was aimed at aerial combat problems that fall within this class. It is useful to distinguish two sub-categories of (6): the missile vs. missile situation typified by the MIRV-ABM example; and the fighter vs. fighter situation involving the action of human pilots.

In the missile vs. missile case, the dynamical equations can often be approximated by relatively simple kinematic laws because of the high accelerations and short engagement times involved in the duel. This will help to simplify any differential games that are formulated in connection with these problems. Moreover, control strategies are not necessarily the most important consideration here and much of the work in category (5) may be applicable.

The fighter vs. fighter duel, which is the problem of interest here, appears to be the more complex problem. Such duels are often characterized by strenuous maneuvers in three dimensions. Rapid changes in altitude, velocity and, hence, in turn radius

for a given load factor, as well as large variations in lift, thrust and drag are experienced frequently. This renders much of the analysis done earlier in connection with bomber defense (where constant altitude and velocity are assumed) quite inapplicable. In terms of problem complexity, the fact that the opponent is considered to be *intelligent* is perhaps of greater significance. This consideration leads to a kind of sequential decision problem that only recently has begun to receive attention.

The Simulation Approach--Several efforts have been undertaken to produce a three dimensional, two-sided simulation of maneuvering air-combat. The Differential Maneuvering Simulator to be installed at Langley Research Center is one of several noteworthy simulation attempts (Refs.26-29). The value of simulators as an effective research tool for studying complex problems is well-known; but the limitations associated with complete reliance on simulation studies are also clearly understood. Simulation is not economical when it is used in a blind, shotgun manner. Its efficient utilization depends on the ability to identify interesting problem areas and to define critical situations and parameters within these areas. In this regard, analytical results, even for simplified problems, can be most helpful insofar as they serve to define situations that warrant more detailed study through simulation.

The Usefulness of Differential Game Theory--The *raison d'etre* for differential game theory is the study of decision problems of the type encountered in fighter vs. fighter duels. Thus, it is natural to expect that attempts to apply the theory to such problems would be made and this is indeed the case. For example, differential game theory has been used to study the control policies for aircraft during short engagements in which the initial

conditions are reasonably well-defined and range over a small set of values. This kind of study has been carried out at the Air Force Academy[†] and at Rand[††]. It is useful for identifying simple maneuver strategies for human pilots to follow.

Another area in which differential game theory has been useful is the identification of regions in the relative position-velocity state space that are favorable to each aircraft. Parametric studies of the tradeoffs between design parameters of the aircraft and variations of the boundaries of these regions can yield meaningful insights. As noted above, such analyses are also useful in conjunction with simulation studies. The problem of the Homocidal Chauffeur due to Isaacs (Ref.3) is a prime example of this kind of analysis. The efforts of Breakwell and Merz (Ref.20), Meier (Ref.30), and Miller (Ref.31) all fall within this category of investigation. We have also followed this approach somewhat in our examination of the dogfight problem, to be described in Chapter 5.

All of the above studies have dealt with problems that were greatly simplified. None of them have led to general techniques for treating more complicated fighter vs. fighter problems in an efficacious way. This is not surprising in view of the previously mentioned analytic and numerical difficulties associated with differential games. On the contrary, it would have been astonishing if such methods had emerged from these studies. Indeed, from our vantage point, it is difficult to believe that general techniques for the direct solution of differential games will become available in the near future.

[†] Private Communication: B. Morgan.
[††] Private Communication: R. Spicer.

What then are the prospects for solving more complicated fighter vs. fighter combat problems? It seems to us that there exists a possibility for solution that heretofore has not been seriously explored. The approach involves the application of Markov game theory. It is based primarily on the following assumptions: (i) pilots can only resolve the state space to a finite degree of accuracy; and (ii) pilots choose their maneuvers from a finite collection of possible maneuvers. With these assumptions, the air combat problem can be reduced to one of controllable Markov chains. From both an analytical and a computational standpoint, this problem is much simpler than a "corresponding" differential game.

The Markov game approach to aerial combat problems will be described in detail in the next Chapter. Before ending this discussion, however, we should note that the approach is not completely divorced from differential games. In fact, although this is not the viewpoint we prefer, one can think of the technique as a way of obtaining approximate solutions to differential games. We will say more about this later.

3. A COMPUTATIONAL APPROACH TO AERIAL COMBAT PROBLEMS

We have just suggested that an approach to the solution of manned aerial combat problems that involves the application of Markov game theory could prove most fruitful. In this chapter we develop such an approach. We begin by showing how the introduction of two physically motivated assumptions leads to the formulation of the aerial combat problem as a meaningful discretized game. Then, concepts of Markov process and state increment dynamic programming are used to develop a feasible computational scheme for solving this game. Two examples that demonstrate the application of the technique are presented in the next chapter.

Problem Formulation

As we saw in Chapter 1, an aerial combat between two vehicles may be described mathematically as a zero-sum differential game. For ease of reference, we repeat below the equations that serve to define such a game:

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), \underline{v}(t)) \qquad \underline{x}(t_o) = \underline{x}_o \qquad (3.1)$$

$$\underline{u}(t)\epsilon U \ , \ \underline{v}(t)\epsilon V \qquad (3.2)$$

$$J(\underline{u}, \underline{v}) = \int_0^T L(\underline{x}(t))dt \qquad (3.3)$$

$$\underline{x}(T)\epsilon R \ ; \ T = \text{free} \qquad (3.4)$$

Briefly, the components of $\underline{x}(t)$ represent the aircrafts' positions, velocities, etc., at some time t; the game terminates at the first time T that $\underline{x}$ enters a prescribed capture region R; and, the objective of the game is to find control strategies $\underline{u}^*\epsilon U$ and $\underline{v}^*\epsilon V$ that yield a saddlepoint of J.

Most of the work in zero-sum games assumes perfect information. Unfortunately, in manned aerial combat situations the pilots rarely have precise knowledge of the state $\underline{x}(t)$ and must base their control actions on imprecise estimates of the current state of play. We have attempted to include this physical constraint directly in the problem formulation. Our approach is to decompose the state-space $\underline{X}$ (or actually a compactified representation $\overline{X}$ of $\underline{X}$)[†] into a finite number of disjoint blocks $S_0, S_1, \ldots, S_N$. These blocks may be of arbitrary size and shape and satisfy

$$\overline{\underline{X}} = \bigcup_{i=0}^{N} S_i \; ; \; S_0 = R$$

As an example, consider a game played in a horizontal plane with both combatants having constant velocity. Here the blocks S_i could be meaningfully associated with crude position estimates (e.g., clock angles-off and suitable range discretizations)[††].

[†] In order for the decomposition to be finite, $\underline{X}$ can be artificially bounded by letting $||\underline{x}||=C_1$ = arbitrary be a reflecting or an absorbing barrier. In the latter case the game ends whenever $||\underline{x}|| \geq C_1$.

[††] An example of such a game with this type of state-space decomposition is studied in Chapter 4.

We assume that both players know the system state only to within a block S_i; a player cannot discern where the state is within S_i, but only the fact that $\underline{x}(t)$ is somewhere (with, e.g., uniform probability) in the known block. Thus, the original state-space has been discretized and the blocks S_i may be thought of as "states" for a discrete game.

Transitions from block to block, i.e., changes in state, take place under the influence of appropriate control actions. We may also discretize the control spaces in a meaningful way. In particular, we assume that the players' control strategies are constructed from <u>finite</u> sets of "canonical control maneuvers,"

$$U_\alpha = \left\{ \underline{u}^1(\cdot),\ \underline{u}^2(\cdot),\ldots,\underline{u}^\alpha(\cdot) \right\}$$
$$V_\beta = \left\{ \underline{v}^1(\cdot),\ \underline{v}^2(\cdot),\ldots,\underline{v}^\beta(\cdot) \right\}\ . \tag{3.5}$$

These sets can encompass basic control maneuvers such as various g turns, sharp pull-ups or dives. They can also include more specialized or complex maneuvers such as a "scissors" or a "yo-yo". Alternatively, the sets U_α and V_β could be composed of various guidance laws (e.g., direct pursuit, proportional navigation, etc.). In any case, the selection of a particular maneuver by a player corresponds to his choosing a particular "control history" to be employed over some subsequent time interval.

Notice that this form of control discretization is quite different from that ordinarily used in solving optimal control problems numerically. We have not quantized the range of control values. Rather, we have discretized the space of control functions, i.e., we have restricted the players to a finite set of control <u>choices</u> or decisions.

The approach to discretizing the control space was also motivated by physical considerations. It is based on the belief that pilots learn certain "stylized" maneuvers in training for aerial combat. Then, in a battle, a pilot selects a particular maneuver that best serves his goals based on a relatively crude assessment of his situation vis-a-vis his opponent. It is easy to see how the combined control and state space discretizations reflect this situation.

In the next section it is shown how the above assumptions reduce the aerial combat "game" to a problem of controllable Markov chains, i.e., to a discrete Markov game. The solution of the discretized game is a blueprint for optimal play, i.e., a way of deciding which control maneuver is best to use in each of the perceived states S_i (in other words, a pair of optimal <u>strategies</u> or <u>feedback</u> <u>control</u> <u>laws</u>).

The Discretized Game

We are now in a position to represent the aerial combat problem by a game defined over a discretized domain. To accomplish this, let us suppose that a particular pair of maneuvers $(\underline{u}\epsilon U_\alpha, \underline{v}\epsilon V_\beta)$ have been selected by the players. Then, a given state $\underline{x}(t)$ would evolve to some new value under the influence of the dynamic equations of motion (3.1). However, the "actual" state $\underline{x}(t)$ is not known; all that is known is that $\underline{x}(t)\epsilon S_i$. Hence, a particular trajectory cannot be "followed" and the most that one can hope to determine are the probabilities of transitions between blocks, under the action of a particular maneuver pair. We define

$$p_{ij}(\underline{u},\underline{v}) \triangleq \text{probability of a transition from block } S_i \text{ to block } S_j \text{ with maneuvers } (\underline{u},\underline{v}).\dagger$$

It is clear that the transition probabilities will play the same role in the discrete game that the equations of state (3.1) played in the continuous game. Inasmuch as we assume that $S_o = R$ is an absorbing state, we have

$$p_{oi} = \begin{cases} 0 & i \neq 0 \\ 1 & i = 0 \end{cases} \tag{3.6}$$

which is equivalent to the "stopping" rule of (3.4). Only in rare cases will it be possible to determine the p_{ij}'s analytically. However, they can usually be obtained numerically (e.g., by Monte-Carlo methods). A technique for computing the p_{ij}'s will be discussed later.

The formulation of the original problem as a discrete Markov game in which transitions from any given state, S_i, to <u>any</u> other state, S_j, are allowed would require storage of the transition probabilities $p_{ij}(\underline{u},\underline{v})$ for all $i=1,\ldots,N$, $j=1,\ldots,N$ and all $\alpha \cdot \beta$ pairs $\underline{u},\underline{v}$. If N is 1000 and each player has a choice of $\alpha=\beta=5$ possible maneuvers, one would have to store 25×10^6 numbers — an unreasonable requirement for most computer facilities. Fortunately, there is a way of simultaneously reducing the storage requirements and computational demands of a conventional Markov game formulation. The approach is based on the nature of the problem and uses concepts of Larson's "state-increment dynamic programming." (Ref.32)

$\dagger$Alternatively $p_{ij}(\underline{u},\underline{v})$ can be regarded as the fraction of states in block S_i that are driven (evolve) into block S_j.

In particular, we assume that once the players choose their control maneuvers $\underline{u},\underline{v}$ in a given state S_i, they must continue to employ these maneuvers until a change in state (to, say, S_j) occurs; once a state transition is perceived, the players may change to a different maneuver. This is appealing from a physical viewpoint since control decisions are made on the basis of perceived information, which in this case, changes only when $S_i \to S_j$ transitions occur. In other words, a new decision is made based on the perceived outcome of the old decision.

From a mathematical viewpoint this means that, in one stage, state transitions are restricted to only <u>adjacent</u> blocks before the players reoptimize over their sets of allowable maneuvers (see Fig. 1). The net effect is a tremendous reduction in the amount of numbers $p_{ij}(\underline{u},\underline{v})$ that must be stored. For a three-dimensional state space (discretized rectangularly) and the numbers given earlier, we now need to store only $25 \cdot 1000 \cdot 3^3 = 675,000$ numbers (in contrast to the 25 million numbers needed if transitions between all blocks are allowed).

To complete our formulation of a discrete Markov game we must delineate the goals of the game. We begin by defining

$$c_{ij}(\underline{u},\underline{v}) \equiv \text{ the average cost of a transition from } S_i \text{ to } S_j \text{ using the maneuver pair } (\underline{u},\underline{v}).$$

At this point, we needn't relate the c_{ij} to the continuous differential game payoff, but it is possible to do so if desired. Specifically, we can write

$$c_{ij}(\underline{u},\underline{v}) = L(\underline{\bar{x}}_j)\Delta_{ij}(\underline{u},\underline{v}) \tag{3.7}$$
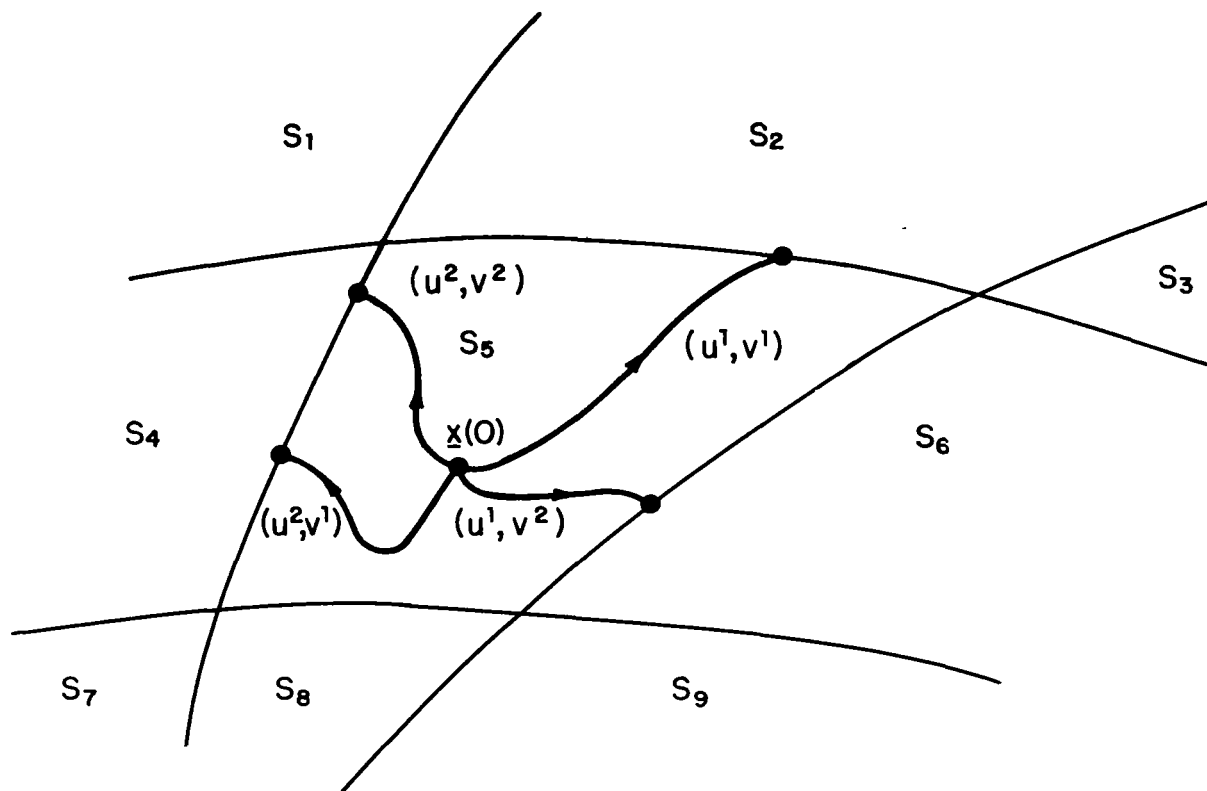
FIG.1   NATURE OF ALLOWABLE STATE TRANSITIONS

where L is the integrand of the payoff of Eq. (3.3), $\bar{x}_j$ is the centroid of block S_j and $\Delta_{ij}(\underline{u},\underline{v})$ is the "average" time of a transition from S_i to S_j using the pair $(\underline{u},\underline{v})$.[†] Then, e.g., if $L \equiv 1$, $c_{ij}(\underline{u},\underline{v})$ is the average time to go from S_i to S_j.

The c_{ij}'s are "one-stage" costs that will accumulate as transitions between discrete states occur. The total cost will depend on the initial state and on the strategies employed by the players. Recall that a strategy corresponds to a control choice for every state of the game. Hence, we define a u-strategy (or policy), π_u, as a set of N control maneuvers (one for each state), i.e., $\pi_u = \{\underline{u}(1),\underline{u}(2),\dots,\underline{u}(N)\}$; $\underline{u}(i)\epsilon U_\alpha$, such that the maneuver $\underline{u}(i)$ is applied whenever the state of the game is S_i.[††] A similar definition is made for π_v. We denote the total cost incurred for a game starting in S_i and played with the policy pair $\pi = (\pi_u,\pi_v)$ by $V^\pi(i)$. Clearly, $V^\pi(i)$ satisfies the equation

$$V^\pi(i) = \sum_{j=1}^{N} p_{ij}(\underline{u}(i),\underline{v}(i))[V^\pi(j) + c_{ij}(\underline{u}(i),\underline{v}(i))] \quad (3.8)$$

Note that the summation need only be taken over blocks S_j adjacent to S_i. The goal of the game is then to find a strategy pair $\pi^* = (\pi_u^*,\pi_v^*)$ such that

$$V^{\pi^*}(i) = \underset{\pi_u}{\text{Min}} \underset{\pi_v}{\text{Max}} \ V^\pi(i) = \underset{\pi_v}{\text{Max}} \underset{\pi_u}{\text{Min}} \ V^\pi(i), \text{ for all } i \quad (3.9)$$

[†]By assuming that maneuvers change only upon a change in perceived state, time becomes, in effect, a dependent variable.

[††]We see that strategies are merely feedback control laws defined over the space $\bar{X}$.

This completes the formulation of the Markov game. To re-capitulate, the relation of this game to the continuous differential game formulated earlier is as follows: the discrete "perceived" states, S_i, replace the continuous state $\underline{x}(t)$; the transition probabilities, $p_{ij}(\underline{u},\underline{v})$, play the role of "equations of motion"; the delineation of S_o and the definition of p_{oi} (Eq.3.6), provide the equivalent termination criterion; and, the objective of finding an optimal strategy pair π^* that satisfies Eq. (3.9) replaces the goal of finding a pair $(\underline{u}^*,\underline{v}^*)$ that provides a saddlepoint of Eq. (3.3). In the next section we discuss methods for solving this discrete Markov game.

Solution of the Markov Game

There has been considerable effort devoted to the analytic study of Markov chains with controllable transition probabilities. Iterative schemes for the solution of single control (one-sided game) problems appear in Howard (Ref.33), Zadeh and Eaton (Ref.34) and Kushner and Kleinman (Ref.35). An excellent study of Markov games appears in Chamberlain (Ref.36) as well as in Kushner and Chamberlain (Ref.37). In this section we present an iterative technique, based on Refs.35-36, that can be used to solve the discrete problem just formulated.

Computation of Transition Probabilities--Before discussing the iterative technique, we indicate the method we use to compute numerical values for the transition probabilities $p_{ij}(\underline{u},\underline{v})$. It is closely related to Monte-Carlo methods. The first stage in this process is a logical sequential ordering of the blocks $\{S_i\}$. For a given pair of maneuvers $(\underline{u},\underline{v})$, M points are uniformly distributed[†] over a given block S_i. Starting at each one, $\underline{x}(0)=\underline{\xi}$,

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[†]Other random distributions of states within a block could alternatively be considered.

of the M points, the equations of motion (3.1) are integrated forward in time until $\underline{x}(\cdot)$ enters an adjacent block, say, S_j. The time

$$\Delta_{min} \leq \Delta_{ij}(\underline{\xi},\underline{u},\underline{v}) \leq \Delta_{max} \qquad (3.10)$$

for this transition is recorded. Δ_{min} and Δ_{max} bound the block transition times, and are generally dictated by the physics of the problem. Note that $\Delta_{ij}(\underline{\xi},\underline{u},\underline{v}) = \Delta_{max}$ only if $\underline{x}(\cdot)$ has not left block S_i.

We let M_j be the number of points initially within S_i that are driven to S_j (with the given strategies $\underline{u}$ and $\underline{v}$). Then

$$p_{ij}(\underline{u},\underline{v}) \approx M_j/M \qquad (3.11a)$$

and

$$\Delta_{ij}(\underline{u},\underline{v}) \approx \sum_{\underline{\xi}\epsilon S_i} \Delta_{ij}(\underline{\xi},\underline{u},\underline{v})/M_j \qquad (3.11b)$$

$$= \text{average transition time from } S_i \text{ to } S_j$$

Finally, $c_{ij}(\underline{u},\underline{v})$ is calculated (e.g., as in Eq. (3.7)). The quantities c_{ij} and p_{ij} for all pairs $(\underline{u},\underline{v})$ are placed in computer storage.[†]

Iterative Technique for Game Solution--The game situation is now specified numerically in terms of the transition probabilities and the "rewards" c_{ij}. The original system equations plus all

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[†]Actually one need not store all the c_{ij}'s but only the N numbers $C_i = \sum_{j} c_{ij}p_{ij}.$

stochastic and nonlinear effects of the problem are absorbed, in essence, by the $p_{ij}(\underline{u},\underline{v})$. The original cost functional is characterized by the $c_{ij}(\underline{u},\underline{v})$. Note that p_{ij} does not depend on the cost functional used.

To apply an iterative scheme to the problem, we first assume that there exists a pursuer policy $\hat{\pi}_u$ such that, for any evader policy π_v, there exists a finite probability that the capture region S_o can be reached in a fixed number (N) of stages, regardless of the policy π_v chosen. This guarantees that $V^{\pi^*}(i)$ will be finite for all i.[†]

Recall that the optimal policy $\pi^* = (\pi_u^*, \pi_v^*)$ and the optimal cost $V^*(i)$ satisfy Eqs.(3.8)-(3.9), i.e.,

$$V^*(i) = \min_{\underline{u}\epsilon U_\alpha} \max_{\underline{v}\epsilon V_\beta} \left\{ \sum_{j=1}^{N} p_{ij}(\underline{u},\underline{v})[V^*(j) + c_{ij}(\underline{u},\underline{v})] \right\} \quad (3.12)$$

A straightforward method of solving for $V^*(i)$ is given by Chamberlain (Ref.36). The method is based on the following:

Theorem 1: If, for any N-vector $\underline{r}$

$$\min \max \sum_j p_{ij}(u,v)[r_j + c_{ij}(u,v)] = \max \min \sum_j p_{ij}(u,v)$$

$$[r_j + c_{ij}(u,v)] \quad (3.13)$$

[†]An alternate approach is to artificially bound $V^\pi(i)$ to be less than some fixed constant V_{max}.

then

a. There is an optimal policy pair $\pi^* = (\pi_u^*, \pi_v^*)$ such that

$$V^{(\pi_u^*, \pi_v)} \leq V^{(\pi_u^*, \pi_v^*)} \left\{ \leq V^{(\pi_u, \pi_v^*)} \right\} \tag{3.14}$$

b. The value of the game $V^* = V^{(\pi_u^*, \pi_v^*)}$ is the unique limit of the sequence

$$V^n(i) = \min_u \max_v \sum_{j=1}^N p_{ij}(u,v)[V^{n-1}(j) + c_{ij}(u,v)]$$

$$\left\{ = \max_v \min_u \sum_{j=1}^N p_{ij}(u,v)[V^{n-1}(j) + c_{ij}(u,v)] \right\} \tag{3.15}$$

We remark that if condition (3.13) is not satisfied (i.e., a saddlepoint cannot be guaranteed) then the terms in braces in Eqs. (3.14)-(3.15) are omitted. In this case $V^*(i)$ is the value of the majorant game; the resultant policies (π_u^*, π_v^*) are the majorant policies.

The iterative scheme suggested by Eq. (3.15) bears a close similarity to the Jacobi method for solving the linear systems of simultaneous equations

$$V^*(i) = \sum_{j=1}^N p_{ij} V^*(j) + c_i \tag{3.16}$$

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namely, the algorithm

$$V^n(i) = \sum_{j=1}^{N} p_{ij} V^{n-1}(j) + C_i \; ; \; V^0 = \text{arbitrary} \qquad (3.17)$$

It is well-known that if the Jacobi iterations converge, then the Gauss-Seidel iterations defined by

$$\widetilde{V}^n(i) = \sum_{j=1}^{i-1} p_{ij} \, \widetilde{V}^n(j) + \sum_{j=i}^{N} p_{ij} \, \widetilde{V}^{n-1}(j) + C_i$$

$$(3:18)$$

will generally converge faster (to V*) than will Eq. (3.17).

This fact was first exploited by Kushner and Kleinman (Ref.35) to improve the convergence rate of solutions to Markov control processes (one-sided games). The extension to Markov games was studied by Chamberlain (Ref.36). The main result is

<u>Theorem 2</u>: If the conditions of Theorem 1 are satisfied, then

a. The sequence $\widetilde{V}^n(i)$ defined by

$$\widetilde{V}^n(i) = \min_{u} \max_{v} \left\{ \sum_{j=1}^{i-1} p_{ij}(u,v)\widetilde{V}^n(j) + \sum_{j=i}^{N} p_{ij}(u,v) \, \widetilde{V}^{n-1}(j) \right.$$

$$\left. + \sum_{j=1}^{N} p_{ij}(u,v) c_{ij}(u,v) \right\} \qquad (3.19)$$

converges to V*(i) for any $\widetilde{V}^0(i)$ and all i.

b.   If $\tilde{V}^0(i) = 0$ for all i then

$$0 \leq V^n(i) \leq \tilde{V}^n(i) \qquad\qquad (3.20)$$

and $V^n(i)$, $\tilde{V}^n(i)$ converge monotonically to $V^*(i)$.

Thus, the iterative scheme embodied in Eq. (3.19) is to be preferred over that of Eq. (3.15). One is guaranteed of a rate of convergence greater than or equal to the iterations of Theorem 1. In addition, the scheme of Eq. (3.19) requires less computer storage since the values $\tilde{V}^n(i)$ are updated as soon as they are obtained, rather than waiting until all blocks have been processed.

Summary

In this section we have presented a different conceptualization of the air-to-air combat problem. Our approach was based primarily on two physical concepts, one relating to the available information sets and the other relating to the nature of the control decisions. This approach allows us to view the original aerial combat problem as a discrete Markov game, with all stochastic and nonlinear effects being absorbed in the transition probabilities.

A relatively new numerical method, based on the Gauss-Siedel technique for solving linear equations, provides a convergent algorithm for solving the problem. The resulting solution will provide the optimal strategies for each player.

Our procedure of applying Markov theory is different from most other applications of Markov processes to continuous control problems. We transform the original physical situation directly

into the context of a discrete Markov game. We maintain that when physical constraints are incorporated, it is this latter problem that is more basic and more closely represents the actual combat situation. Other approaches (Refs.35 through 37) first apply Hamilton-Jacobi theory to the original continuous problem. They then solve the resultant Hamilton-Jacobi partial differential equation by a particular discretation scheme that bears an analogy with Markov games. However, no Markov "game" is actually formulated.

In the following chapter, we apply the theory and the associated computational scheme to study two classic "differential games".

4. TWO EXAMPLES

In this chapter, the theory and computational schemes developed in Chapter 3 are applied to study analogues of two well-known differential game problems: Issacs' classic "Homicidal Chauffeur Problem" and the more complex "Two Car Problem." In each case we discuss the results of our Markov formulation in light of known analytic properties of the solution to the corresponding continuous game.

Homicidal Chauffeur Problem

The Homicidal Chauffeur Problem is described in detail in Issacs (Ref.3) and in Breakwell and Merz (Ref.20). Briefly, a pursuer P (chauffeur) moving at constant speed w_1 with a finite minimum turning radius R, chases an evader E (pedestrian) moving at a lower constant speed w_2 but with an infinitely small turning radius. The game ends (capture) when the mutual distance becomes less than a given capture radius L. The pursuer strives to minimize the time-to-capture while the evader tries to maximize it.

The equations of motion for this problem, written in a pursuer centered system of coordinates are:

$$\dot{x} = -\frac{w_1}{R} y\, u + w_2 \sin v$$

$$\dot{y} = \frac{w_1}{R} x\, u - w_1 + w_2 \cos v \qquad\qquad \left.\begin{array}{c} \\ \\ \end{array}\right\} (4.1)$$

where

(x,y) = position of E relative to P

u = pursuer's rate of turn (P's control), $|u| \le 1$

v = relative angle between velocity vectors of E and P (E's control)

The parameter values used in our study of this problem were

$$w_1 = 1.0 \ , \ w_2 = .7 \ , \ R = 1.5 \ , \ L = 1.0$$

The analytic solution to the above problem is characterized by a set of singular curves in the (x-y) state-space. Using Issacs' terminology, the curves consist of barriers, dispersal curves, universal curves and equivocal lines. Some of these curves are shown in Fig. 2.[†] Of the various singular curves, the barrier appears to be most important from a practical point of view. Under optimal play, no trajectory can cross the barrier and the min-max time to capture is discontinuous across this curve. Moreover, the barrier (along with the equivocal curve) serves to separate regions of different pursuit strategy. Essentially, the optimal pursuit strategy is to turn hard away from the evader whenever E is in the region enclosed by the barrier and hard into E otherwise. [The optimal pursuit control is u=0, (i.e., a straight dash) on the universal line. On the dispersal line u=±1, i.e., either control choice will result in the same optimal cost.] A typical optimal trajectory ABCD is also shown in Fig. 2. This trajectory is based on E using his *penetrating strategy* [Ref.3] on the equivocal line. At point B the pursuer switches from a hard turn left (u=-1) to a hard turn right (u=+1) and he switches to a dash (u=0) upon intersecting the universal line at C.

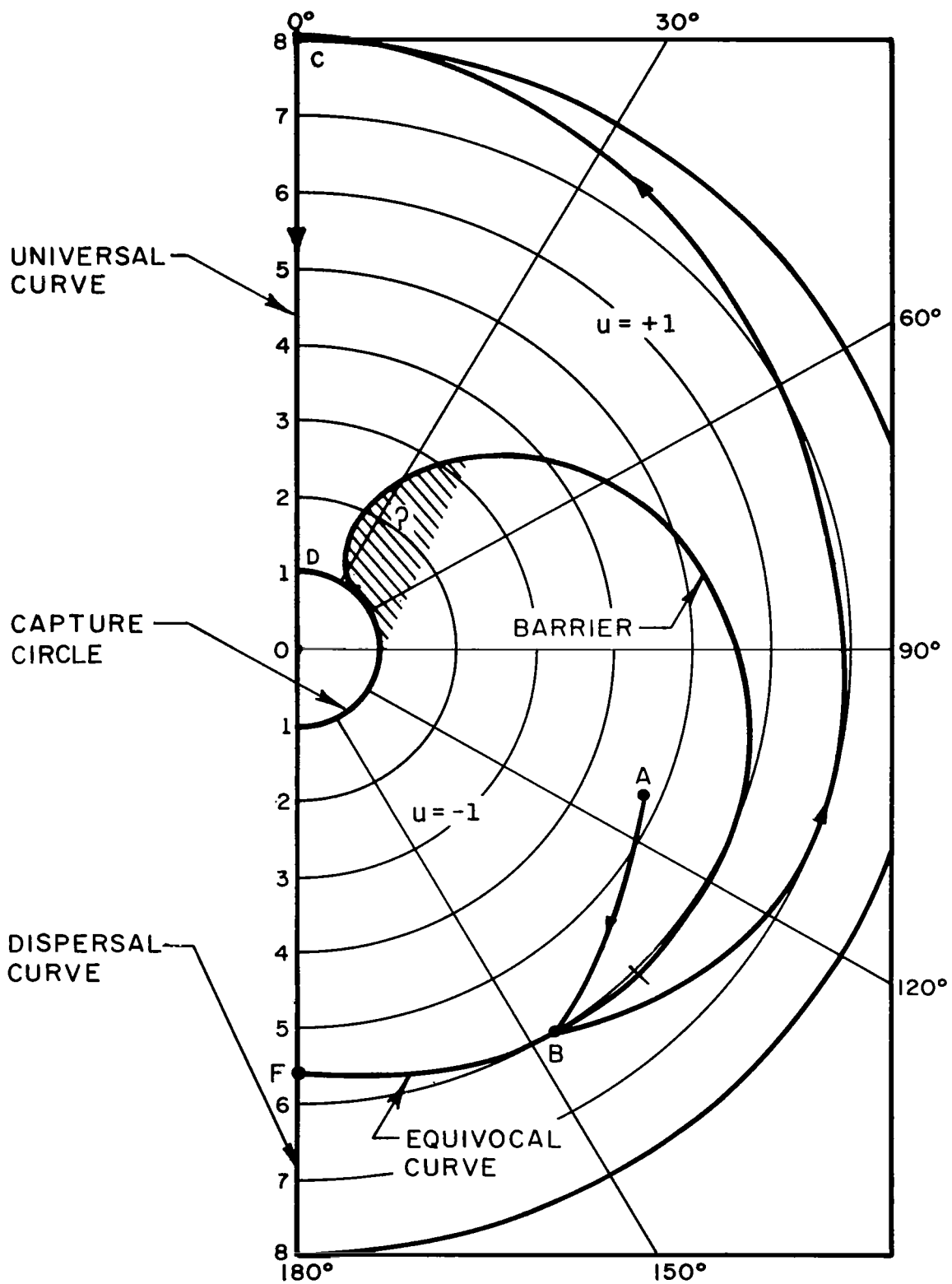[†]Since the game is symmetric with respect to the line x=0, the solution for $x \ge 0$ is shown only.

FIG.2   SOLUTION TO ISSACS' HOMICIDAL CHAUFFEUR PROBLEM
$w_1 = 1.0$, $w_2 = .7$, R=1.5, L=1

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Unfortunately, the above is not the _complete_ analytic solution to the Homicidal Chauffeur problem. In the shaded region of Fig. 2 the solution is not completely known. Breakwell and Merz (Ref.20) have spent considerable effort towards understanding the nature of the solution in this region. They found numerous esoteric switch curves and equivocal curves with which to characterize the optimal solution, yet the solution is still not completed.

A Computational Approach to the Homicidal Chauffeur Problem

Problem Reformulation--Our characterization of manned combat problems, coupled with the computational framework that we have developed, was applied to the above problem. We decided to decompose the state-space $\underline{X}$ into blocks S_i with respect to the polar coordinates (r,θ) where

$$r = (x^2+y^2)^{1/2}$$
$$\theta = \tan^{-1}(y/x)$$

(4.2)

These coordinates seem most natural for a problem of this type played in the plane.

In order to bound the state-space, a reflecting barrier was imposed at $r=10$, so only the region $r\leq 10$ was considered. In addition, by problem symmetry, only the region $0\leq\theta\leq 180°$ was investigated, and a reflecting barrier was also set along the y-axis. Thus

$$\underline{X} = \{(r,\theta): \quad r\leq 10, \quad 0\leq\theta\leq 180°\}$$

(4.3)

A 20° discretization of the angular coordinate ($\Delta\theta$) was chosen along with a 1 unit radial discretization (Δr). Thus, there was a total of 81 state blocks plus the capture circle ($R=S_0$) that comprised the (compactified) state-space $\overline{X}$. The state-space decomposition, along with our particular ordering of the blocks is shown in Fig. 3.

The next step in our analysis was the specification of the sets of "canonical maneuvers." For the pursuer we chose minimum radius left and right turns and a straight ahead dash; so, $u=-1,+1,0$, respectively. (Intermediate maneuvers, e.g., a 2R turn, could be included if desired.) The evader, on the other hand, can change his direction instantaneously. Thus, he chooses the angle v between velocity vectors w_1 and w_2. We constructed a finite set of evader controls by restricting $v(t)$ to take only the values $n\pi/4$, $n=1,3,5,7$. Hence E has a total of four different "maneuvers" from which to choose his control.

Given the state-space decomposition and the particular choice of control sets, the transition probabilities $p_{ij}(u,v)$ were computed for all 81 blocks S_i for each of the 12 different pairs of control strategies. A total of 36 points were uniformly distributed over each block S_i. For a given strategy pair (u,v) the equations of motion (4.1) were integrated starting from each initial point.[†] The integrations were stopped when the resulting trajectory left S_i, or met a reflecting barrier. The fraction of points that entered S_j was taken as $p_{ij}(u,v)$. Since transitions are allowed only to adjacent blocks, a maximum of 9 nonzero numbers need be computed and stored for each block S_i.

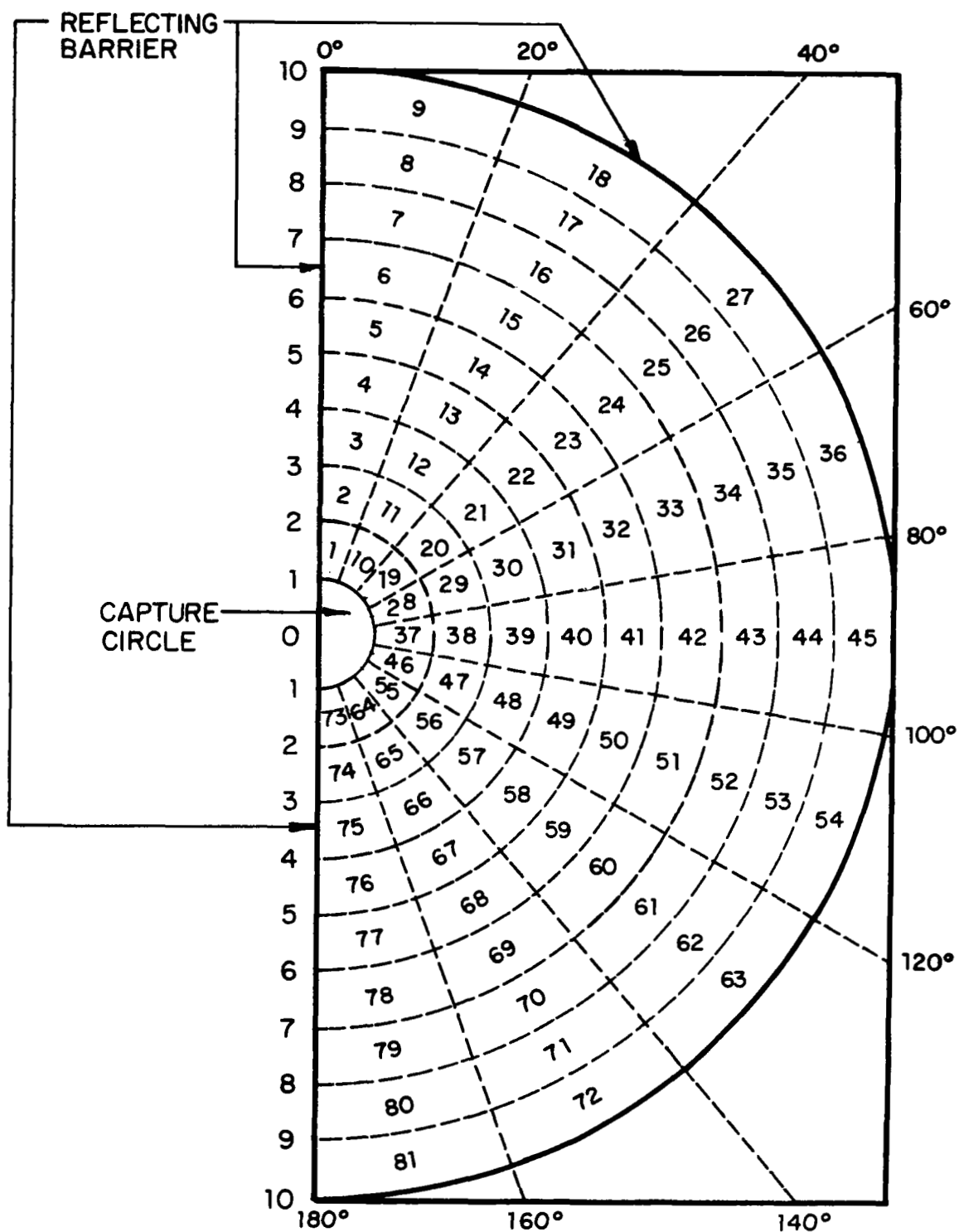[†]We chose $\Delta_{min} = .1$, $\Delta_{max} = 4.0$.

FIG.3   STATE-SPACE DECOMPOSITION AND ORDERING FOR
         HOMICIDAL CHAUFFEUR PROBLEM

For simplicity and computational ease, the quantities $c_{ij}(u,v)$ were all set equal to 1. Thus, the payoff for the Markov game is the "number of block transitions to capture," as contrasted with the "time to capture" payoff of the original Homicidal Chauffeur problem.

Computational Solution--Having formulated a well-defined discrete Markov game in terms of the transition probabilities p_{ij} and costs c_{ij}, the computational algorithm of Theorem 2 was applied. We sought the majorant policy pair (i.e., min-max solution) since we did not a priori verify the existence of a saddle-point. The algorithm converged monotonically to V^*, (π_u^*, π_v^*) in about 60 iterations, starting at $V^o(i) = 0$ for all i.

The resultant discrete solution is shown in Fig. 4. The number within each block is the optimal cost $V^*(i)$. For blocks in the region enclosed by the solid curve it is optimal for the pursuer to turn away (u=-1) or flee (u=0) from the evader. Elsewhere P turns into E (u=+1). Similar results hold by symmetry for $180° \leq \theta < 360°$. Note that this strategy will cause the resultant state trajectory to "chatter" back and forth across the y-axis prior to capture. This is a result of the pursuer's knowing E's position only to within a block; hence, the disappearance of the "universal line".

A comparison of the discrete solution with that of the corresponding continuous game is also found in Fig. 4. This comparison is somewhat tenuous because of the two different cost functionals used, i.e., time to capture vs. block transitions to capture. More importantly, however, we have not attempted to approximate the continuous game solution. We have posed and solved a basically different, but related problem — one that we feel is a
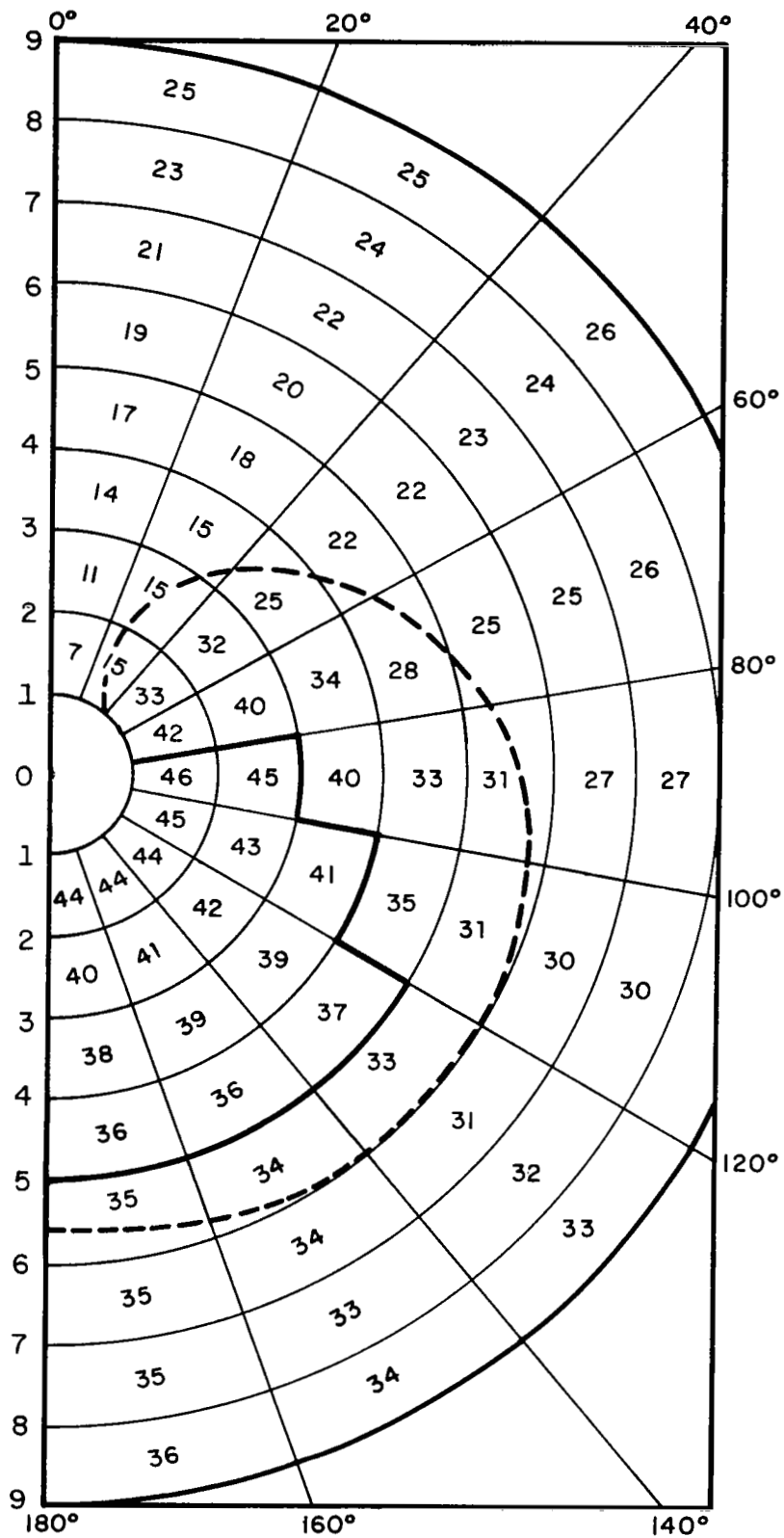
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FIG.4   OPTIMAL DISCRETE SOLUTION TO HOMICIDAL CHAUFFEUR PROBLEM

more meaningful description of a physical situation. Nevertheless, there are some conclusions to be drawn from such a discrete-continuous comparison.

First, the inclusion of the discretized u=-1,0 region within the corresponding region (dashed curve) for the continuous game can be explained largely on the basis of the degradation of P and E's position estimates. P's lack of perfect position information causes him to adopt a more conservative strategy to avoid the heavy penalties associated with making an incorrect decision. For example, if E is located "above" the dashed curve P should definitely turn hard right, or else the evader could be driven into a region of very high cost. On the other hand if E is "below" this curve, the state is already in a region of high cost and the resulting difference between u=-1 or +1 is not so pronounced. Since the pursuer cannot distinguish with certainty on which side of the curve E lies, the more conservative strategy is to apply u=+1 in this vicinity. This is entirely consistent with "minimax" notions and is a most interesting phenomena.

The evader on the other hand does not have a precise estimate of P's position. This will result in E's not being able to direct his velocity precisely away from P. Therefore, there arises some probability that if u=+1, E could be driven unwittingly across the dashed curve into a region of low cost, i.e., the "barrier" has become "porous". Thus P can capitalize on evader error in this region by applying u=+1 and there results an additional "lowering" of the strategy barrier.

There is a second interesting comparison to be found in Fig. 4. In the continuous game, the optimal cost is sharply discontinuous across the barrier. Shown within each block of Fig. 4

is the optimal cost (average number of stages to capture) for the discrete game. Notice how V*(i) strongly varies in the vicinity of the dashed curve. It therefore appears that there is a "discrete, smoothed discontinuity" in this region. This phenomena can be illustrated further by plotting the block cost along a radial segment for fixed θ. Figure 5 shows three of these stepwise cost profiles. Also shown, for the given θ slice is the radial region through which the continuous barrier passes. Notice that in all cases the continuous barrier passes through a region in the state space where the discrete cost has decreased sharply from block to block and has reached a minimum. From this point on, the cost increases with r in a seemingly smoother manner.

The above demonstrates that it may be possible to delineate the region in the state-space through which the continuous cost barrier passes. Needless to say this would give only a crude approximation in view of our generally crude decomposition of the state space. A finer decomposition could conceivably give a better approximation. Thus, although our approach does not involve the approximation of continuous solutions, it appears that our method does provide insight into some important characteristics of the continuous game.

Two Car Problem

Perhaps the most obvious extension of the Homicidal Chauffeur Problem is the situation in which both players have finite minimum turning radii. This is the so-called "Two Car Problem" and is discussed briefly in Issacs (Ref. 3).
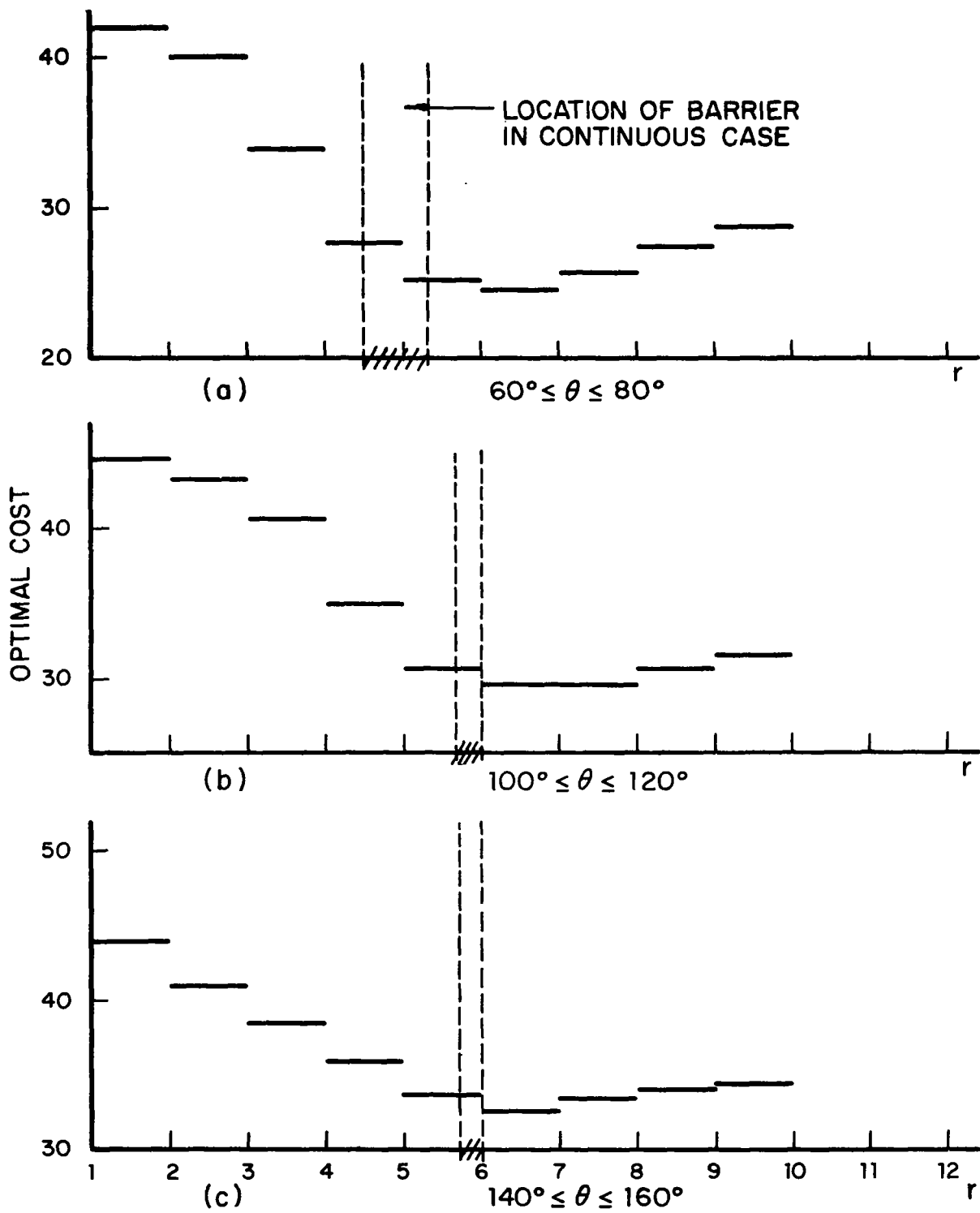
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FIG.5   DISCRETE OPTIMAL COST PROFILES

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The equations of motion for this problem (written with respect to a pursuer-centered system of coordinates) are similar to (4.1) and are given by

$$\dot{x} = -\frac{w_1}{R_1} y \, u + w_2 \sin \psi$$

$$\dot{y} = \frac{w_1}{R_1} x \, u - w_1 + w_2 \cos \psi \qquad\qquad (4.4)$$

$$\dot{\psi} = \frac{w_1}{R_1} u - \frac{w_2}{R_2} v \quad ; \quad |u| \leq 1 \, , \, |v| \leq 1$$

where ψ is the angle between E and P's velocity vectors. R_1, u and R_2, v are, respectively, P and E's minimum turning radius and rate of turn.

Thus, the two-car game is played in the three dimensional (x, y, ψ) state space and the geometric constructions that one uses in two-state problems become rather awkward to apply here. Therefore, it is not surprising that little is known regarding the analytic solution to the two car problem. Issacs' (Ref.3) gives the BUP (Boundary of Useable Part) and portions of the semi-permeable surface (barrier) emanating from the BUP. Miller (Ref.31) has numerically obtained a family of curves that give parameter values for which the entire space is capturable (i.e., for which the evader cannot escape). There is little in the literature beyond these results.

We began our study of the two-car problem by choosing a set of parameter values for which capture was assured for all states. The curves given in Ref.31 were used as a guide; the parameters chosen were

$$w_1 = 1.0 \; , \; w_2 = .8 \; , \; R_1 = 1.5 \; , \; R_2 = .5 \; , \; L = 1.0$$

Computational Solution to Two-Car Problem

As in the Homicidal Chauffeur Problem, we began our reformulation of the two-car problem by choosing a meaningful state-space decomposition. Decomposition of the x-y plane with respect to r-θ polar coordinates was again chosen. This time the θ coordinate was discretized according to clock positions (i.e., every 30°). Thus, associated with clock position $j(1 \leq j \leq 12)$ we have the angular sector

$$30j-15° \; \leq \; \theta \; \leq \; 30j+15°$$

i.e., a given state block subtends an angle of 30°, centered at a clock position.[†]

The radial coordinate r was discretized in 1 unit intervals as before. A reflecting barrier was placed at r=10 to bound the state space.

The ψ coordinate was discretized in a manner that was motivated by human estimation capabilities. A precise estimation of relative velocity direction is generally quite difficult to obtain visually. Accordingly, the ψ coordinate was decomposed into only four segments:

[†]In the Homicidal Chauffeur Problem there is a universal line and a dispersal line at the 12 and 6 o'clock positions respectively. We centered our blocks at clock positions in order to see whether the existence of a universal/dispersal curve would be reflected in suitable properties of the discrete solution.

a) $45° < \psi \le 135°$: E moving from "left to right" relative to P

b) $135° < \psi \le 225°$: E and P moving in "opposite" directions

c) $225° < \psi \le 315°$: E moving from "right to left" relative to P

d) $-45° \quad \_ \quad 45°$: E and P moving in the "same" direction

Thus, the state-space $\underline{X}$ was decomposed into $9 \cdot 12 \cdot 4 = 432$ blocks S_i plus the capture region

$$S_o = \{(r,\theta,\psi) : \quad r \le 1\}$$

We felt that the resultant r-θ-ψ decomposition was a reasonable representation of the information sets available in an actual encounter between two manned vehicles.

For the sets of canonical maneuvers we chose only the extreme cases of: (a) hard turn right, (b) hard turn left, and (c) straight dash. Thus, pursuer and evader each had a total of three control maneuvers, characterized by $u,v = -1,+1,0$ respectively.

Having decomposed the state space in a meaningful manner and having assigned the sets of canonical maneuvers, the transition probabilities $p_{ij}(u,v)$ were computed for all i,j and pairs (u,v).[†]

[†]A total of 27 points per block was used (three points along each dimension) with $\Delta_{min}=.1$, $\Delta_{max}=2.0$. Each point was equidistant from any adjacent point.

Since transitions were allowed only to adjacent blocks, there was a maximum of 27 non-zero numbers p_{ij} associated with a given i. The resultant total of $432 \cdot 27 \cdot 9 \simeq 10^5$ numbers was efficiently stored on a rapid-access computer file.[†]

Once again, for simplicity, the rewards $c_{ij}(u,v)$ were all set equal to unity. Thus, the original cost of "time to capture" was replaced by "stages to capture".

The resultant discrete Markov game that we formulated was solved using the algorithm of Theorem 2. Only the min-max solution was obtained for π_u^*, π_v^*, $V^*(i)$. The computed pursuer strategy is shown in Fig. 6. It appears that P's strategy is basically to turn into E, although there are a few blocks where this is not the case. In particular note that (except for a few blocks) the pursuer's optimal strategy for blocks centered at the 12 o'clock position is $u = 0$ (i.e. straight dash). To the left of 12 o'clock $u = -1$ is optimal while $u = +1$ is optimal when E is located to the right. Thus trajectories tend to converge into the forward $u = 0$ region which can be called a <u>universal zone</u> (corresponding to a similar phenomenon in the continuous case). Also note in Fig. 6 the existence of a region at the 6 o'clock position where the pursuer can apply $u = \pm 1$ with equal resultant cost. This is expected by problem symmetry and is reminiscent of the phenomena associated with dispersal curves in continuous games. Hence, this region may be regarded as a <u>dispersal zone</u>.

Unfortunately, a thorough interpretation of these results requires an understanding of how the resulting state trajectory

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[†] A conventional dynamic programming formulation of the Markov game would require $(432)^2 \cdot 9 \simeq 1.7 \times 10^6$ such transition probabilities.
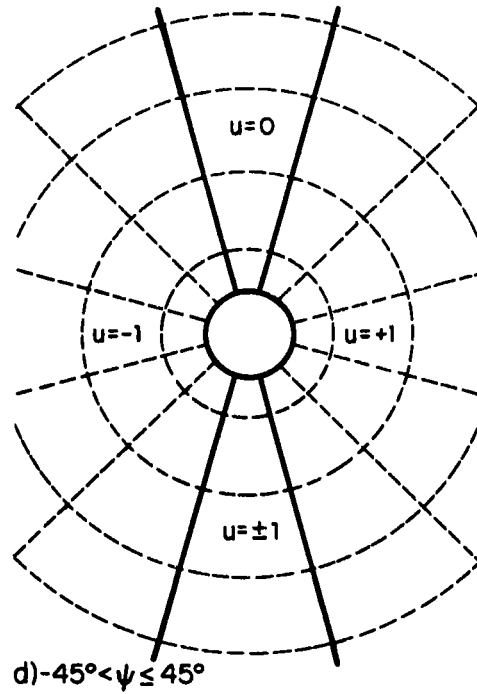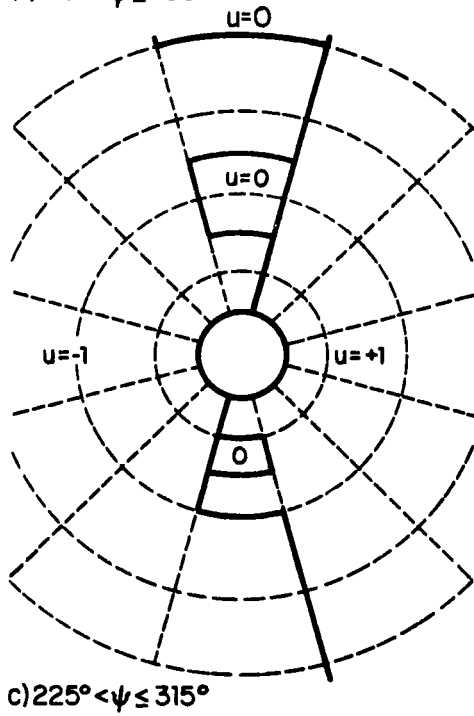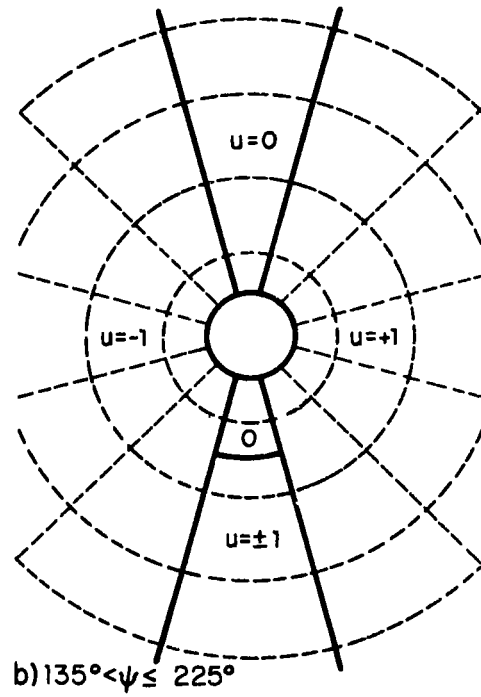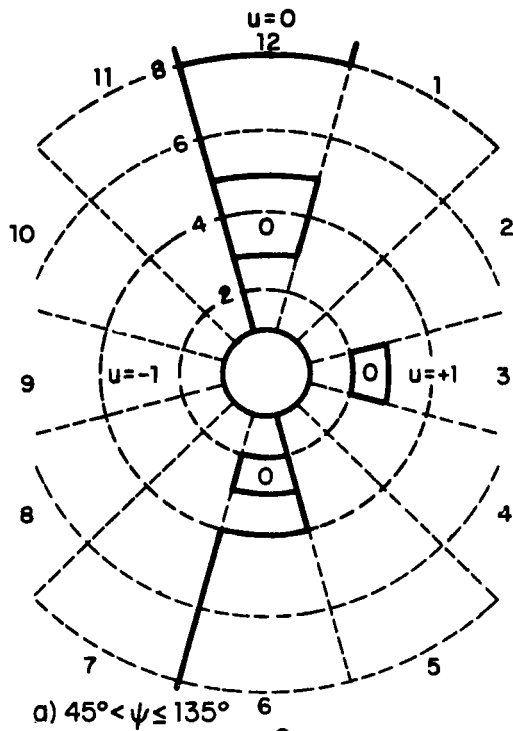
FIG.6   PURSUERS OPTIMAL DISCRETE FEEDBACK STRATEGY, TWO CAR
        PROBLEM: $w_1 = 1.0$, $w_2 = .8$, $R_1 = 1.5$, $R_2 = .5$, $L = 1$

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evolves in the three dimensional (r,θ,ψ) space. Because the game is not symmetric with respect to the plane $y = 0$ (recall that there was symmetry with respect to $y = 0$ in the Homicidal Chauffeur problem) the nature of state evolution may be complex. A study of this type has been left for further research.

The optimal costs V* for the 432 blocks are presented in Table 2. The blocks are indexed by the triple (i,j,k) $1 \leq i \leq 9$, $1 \leq j \leq 12$, $1 \leq k \leq 4$. Thus block $S_{\alpha\beta\gamma}$ includes the region

$$S_{\alpha\beta\gamma} = \{(r,\theta,\psi): \alpha < r \leq 1+\alpha, \ 30\beta - 15° < \theta \leq 30\beta + 15°,$$
$$90\gamma - 45° < \psi \leq 90\gamma + 45°\}$$

Notice that the smallest cost occurs for block $S_{1,12,2}$ i.e. where the evader is directly in front of P and on a collision course. Notice also that there is a symmetry here, namely the costs are equal for the following corresponding blocks.

$$S_{ij2} \sim S_{1,12-j,2}$$

$$S_{ij4} \sim S_{1,12-j,4} \qquad \text{for all } i,j$$

$$S_{ij1} \sim S_{1,12-j,3}$$

This symmetry is entirely expected from physical reasoning.

In the study of the Homicidal Chauffeur problem we saw that the continuous cost barrier was located approximately in a region of the state-space where the discrete cost was at a "valley". This fact, coupled with the data in Table 2, suggest that it may be possible to demarcate a region in the state-space through which the continuous barrier passes. This was done by plotting

TABLE 2

Optimal Costs for the Two-Car Markov Game

| k	Clock Position j	Radial Position i								
		1	2	3	4	5	6	7	8	9
1	1	23.06	23.44	26.66	30.33	34.92	39.33	43.13	46.93	50.05
	2	53.83	51.77	42.87	36.41	37.51	41.95	45.66	49.44	51.94
	3	59.54	56.43	49.22	41.43	41.24	44.60	48.32	51.66	53.70
	4	59.24	56.60	52.40	46.95	48.27	51.06	53.77	55.16	56.09
	5	61.37	58.71	55.68	52.22	53.33	55.52	57.17	58.16	58.54
	6	61.51	58.92	57.15	55.81	57.21	58.78	59.81	60.32	60.48
	7	60.86	57.36	55.74	54.52	54.53	56.22	57.57	58.61	58.90
	8	59.56	57.39	53.95	49.41	49.11	50.36	52.31	54.31	55.61
	9	49.50	55.87	51.60	46.40	43.05	44.25	46.61	49.82	52.39
	10	27.94	40.44	40.79	37.68	36.88	38.78	41.90	45.52	48.96
	11	12.32	19.92	24.77	28.93	32.80	37.00	40.98	44.74	48.09
	12	13.69	21.04	25.43	29.44	33.71	38.03	42.01	45.72	48.89
2	1	18.68	21.88	24.94	28.81	32.57	36.70	40.69	44.35	47.78
	2	50.64	51.88	44.29	38.77	37.98	41.47	44.25	47.91	50.79
	3	59.88	56.94	51.21	45.34	41.91	44.89	48.21	51.64	53.66
	4	60.69	58.27	54.40	48.70	48.82	51.11	54.10	55.96	56.77
	5	60.71	58.15	55.68	53.60	54.63	56.47	57.72	58.79	59.10
	6	60.24	58.70	57.62	57.86	59.14	60.26	60.89	61.06	60.73
	7	59.92	57.48	55.09	53.05	54.07	55.90	57.15	58.19	58.49
	8	60.13	57.80	53.95	48.26	48.40	50.71	53.70	55.53	56.31
	9	59.56	56.65	50.97	45.14	41.65	44.60	47.98	51.38	53.41
	10	50.56	51.80	44.20	38.68	37.86	41.34	44.10	47.76	50.63
	11	18.69	21.87	24.90	28.77	32.53	36.64	40.62	44.28	47.72
	12	8.18	14.92	20.47	25.28	29.48	33.30	37.17	40.96	44.50
3	1	12.28	19.87	24.74	28.92	32.78	36.96	40.94	44.72	48.07
	2	28.10	40.65	40.99	37.85	37.04	38.95	42.08	45.61	49.04
	3	49.99	56.38	52.04	46.78	43.41	44.61	46.93	50.14	52.71
	4	60.37	58.08	54.59	49.98	49.66	50.88	52.80	54.82	56.13
	5	62.09	58.34	56.50	55.24	55.23	56.91	58.26	59.32	59.62
	6	62.03	58.79	57.35	55.23	56.76	58.29	59.28	59.76	60.01
	7	60.22	57.64	54.79	51.45	52.87	55.04	56.64	57.51	57.88
	8	58.57	55.77	51.78	46.45	47.94	50.75	53.48	54.83	55.69
	9	59.41	56.37	49.16	41.37	40.99	44.35	48.20	51.51	53.57
	10	53.95	51.89	42.93	36.41	37.48	41.91	45.62	49.40	51.89
	11	23.17	23.47	26.68	30.35	34.92	39.33	43.13	46.94	50.07
	12	13.65	21.00	25.43	29.45	33.67	37.98	41.96	45.71	48.89
4	1	21.73	23.22	27.07	30.76	35.16	39.62	43.15	46.97	49.99
	2	40.69	35.90	35.37	35.27	37.35	41.72	45.38	49.08	51.93
	3	51.19	50.95	47.48	43.34	41.89	44.34	48.02	51.35	53.45
	4	59.21	56.72	53.07	49.31	48.25	49.87	52.09	54.68	56.38
	5	62.47	58.70	56.71	54.06	53.55	55.14	56.81	58.39	59.18
	6	61.93	59.08	57.40	56.09	56.84	58.30	59.45	60.23	60.54
	7	61.09	57.56	55.76	53.20	52.76	54.37	56.06	57.63	58.40
	8	58.31	55.86	52.28	48.64	47.65	49.29	51.60	54.18	55.87
	9	51.16	50.44	46.95	42.90	41.53	43.99	47.91	51.22	53.32
	10	40.91	36.03	35.18	35.09	37.32	41.69	45.35	49.05	51.89
	11	21.86	23.28	27.10	30.79	35.18	39.65	43.18	47.00	50.03
	12	16.14	21.61	25.69	29.33	33.70	38.14	42.14	45.99	48.93

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cost profiles (as we did in Fig. 5) and noting the regions of rapid cost change. The results of this analysis are shown in Fig. 7. Unfortunately, there does not presently exist an analytic solution to the continuous two-car problem with which to compare our result. A partial description of the barrier is given in Issacs (Ref. 3) and bears some similarity in shape to the shaded region of Fig. 7. Any further comparisons at this point would require more analytic research.

Summary

We have seen how our conceptualization of manned games of conflict can be used to solve existing problems. The results of solving the Homicidal Chauffeur problem show that our approach and technique are basically sound. The application to a three dimensional problem indicates some of the future potential and usefulness of the method.
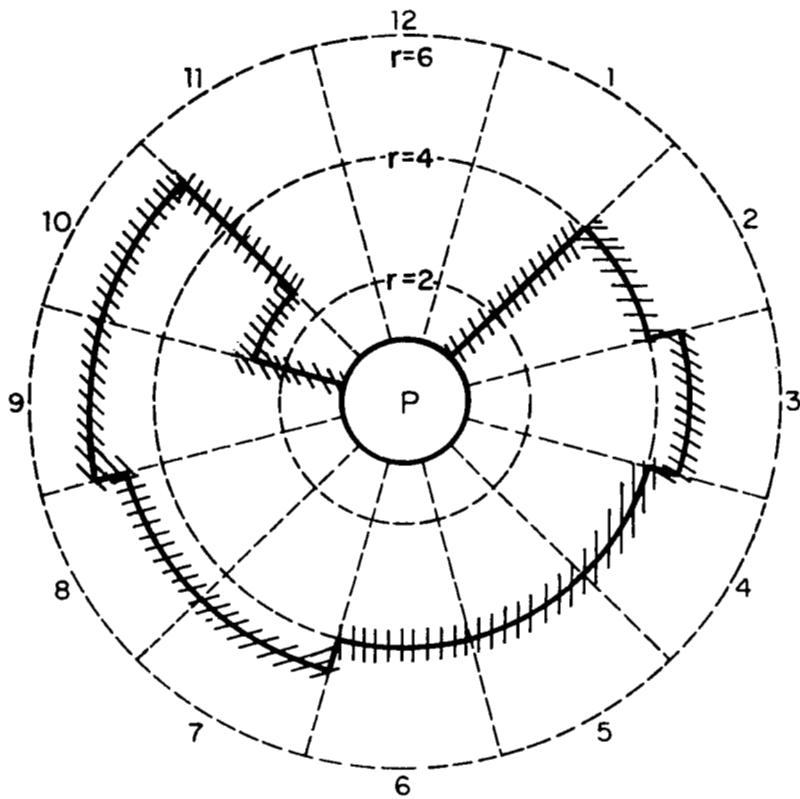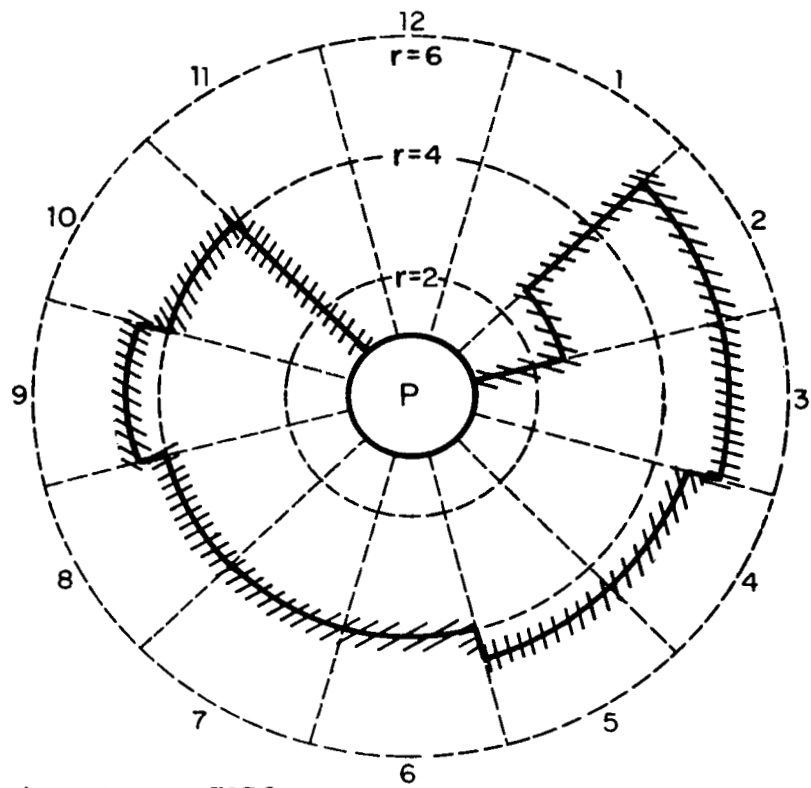
a) $45° < \psi \leq 135°$
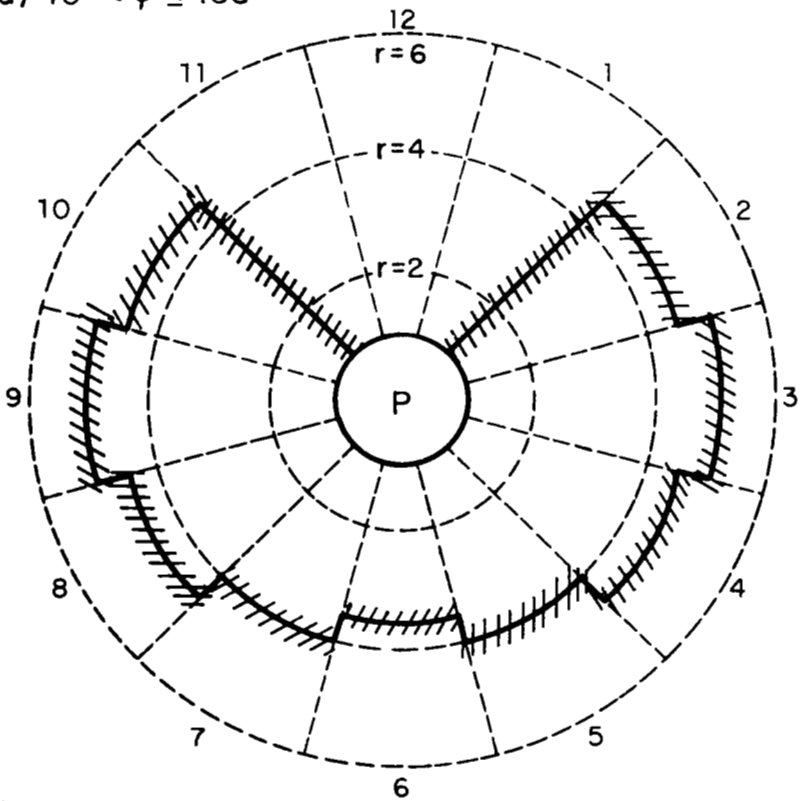
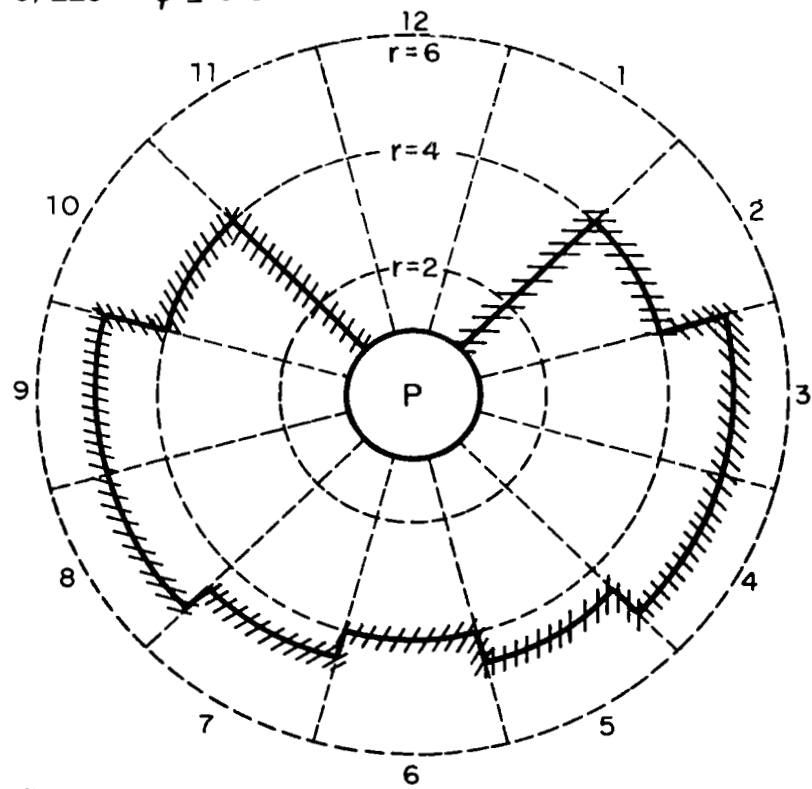b) $135° \leq \psi \leq 225°$

FIG.7   APPROXIMATE LOCATION AND SHAPE OF CONTINUOUS BARRIER:
FROM ANALYSIS OF DISCRETE COST PROFILES

c) $225° < \psi \leq 315°$

d) $-45° < \psi \leq 45°$

FIG.7   (CONTINUED)

5. THE DOGFIGHT PROBLEM

Thus far, we have been concerned primarily with pursuit-
evasion games. These games are important in the study of aerial
combat but they represent a simplification of realistic aerial
duels, or "dogfights". In this chapter, we consider the "dog-
fight" problem. We begin with a general discussion of the problem
and develop possible approaches to it. Then, we examine a "simple"
example in order to illustrate the ideas involved and to indicate
the difficulties inherent in the problem.

Some General Considerations

In a pursuit-evasion game, one player (the pursuer) attempts
to destroy or capture his opponent (the evader) whose goal is to
avoid this outcome. The situation in a realistic aerial duel, or
"dogfight", is more complicated in that each participant wishes
to destroy his opponent and each wishes to avoid extermination.
Nonetheless, it is possible to formulate a differential game that
embodies the essential elements of a dogfight. Let us see how this
might be done.

There are four possible outcomes in a dogfight between two
vehicles (say, A and B). These are: (i) A destroys B; (ii) B
destroys A; (iii) A and B are destroyed; and (iv) neither A nor B
is destroyed. With each of these "outcomes" we can associate a
termination criterion for the game. For example, consider the
two vehicles as point masses moving in a plane. Let each vehicle
have a region $R_{(.)}$ (of, say, weapons effectiveness) attached to
it; the orientation of the region may depend on the direction of
the velocity vector of the corresponding vehicle, e.g., as illus-
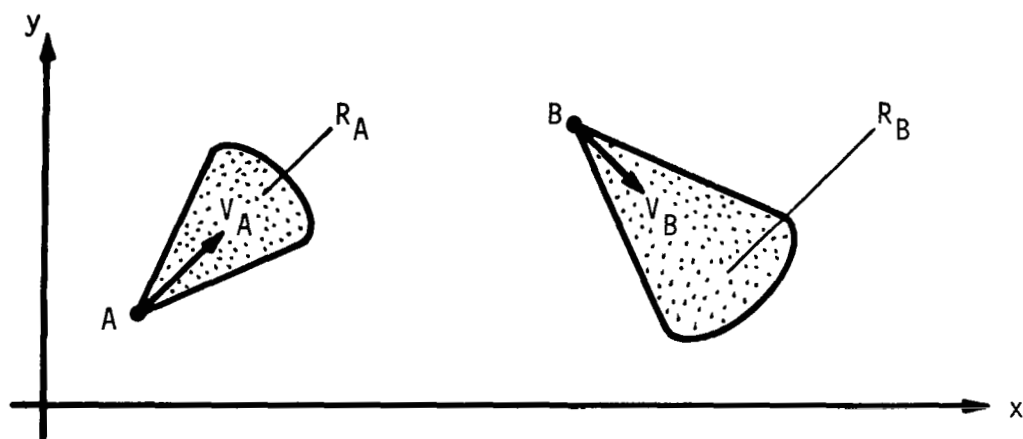trated in Figure 8. Suppose that a vehicle is destroyed the

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FIG.8   SPATIAL ORIENTATION OF TWO VEHICLES WITH
         REGIONS OF COMBAT EFFECTIVENESS.

moment it enters the region of weapons effectiveness of his opponent. Then, the termination criterion corresponding to outcome (i) is that B enter region $R_A (B \epsilon R_A)$. Similarly, outcomes (ii) and (iii) are characterized by $(A \epsilon R_B)$ and $(B \epsilon R_A$ and $A \epsilon R_B)$, respectively. Finally, the fourth outcome is characterized by conditions that A never enters R_B and B never enters R_A for the duration of the game.

More generally, let $\underline{x}$ denote the state of the game (it evolves according to some prescribed differential equation) and let ψ_A and ψ_B be the sets of states corresponding to outcomes 1 and 2, respectively. Then the four "outcomes" of the game are described by the following terminal constraints[*]

1. $\underline{x} \, \epsilon \, \psi_A$: (A destroys B)

2. $\underline{x} \, \epsilon \, \psi_B$: (B destroys A)

3. $\underline{x} \, \epsilon \, \psi_A \cap \psi_B = \psi_{AB}$: (Both A and B destroyed)

4. $\underline{x} \, \epsilon \, \overline{\psi_A \cup \psi_B} = \psi_D$: (Neither A or B destroyed).

The game terminates the first time $\underline{x}$ enters either ψ_A, ψ_B or ψ_{AB}. If it never enters any of these sets (i.e., $x \epsilon \psi_D$) during the entire play of the game, we have a "draw."

---

[*] $\psi_A \cap \psi_B$ and $\psi_A \cup \psi_B$ denote the intersection and union of ψ_A and ψ_B, respectively, and the "bar" indicates the complement of the "barred" set. Thus, $\underline{x} \epsilon \psi_A \cap \psi_B$ means that $\underline{x}$ is in both ψ_A and ψ_B, and $\underline{x} \epsilon \overline{\psi_A \cup \psi_B}$ means that $\underline{x}$ is not in ψ_A or ψ_B.

Clearly, for player A, outcome 1 is most preferable and outcome 2 least preferable. The preference ordering of outcomes 3 and 4 is not obvious. Undoubtedly, most pilots would prefer the fourth outcome but one can easily envision situations in which the third outcome is preferable from an overall strategic point of view. For illustrative purposes let us assume that player A prefers mutual destruction to mutual escape and that player B has the same preference. Then, the dogfight problem can be formulated as a nonzero sum game with, for example, the following terminal payoff functions

$$\underline{x} \varepsilon \psi_A \qquad J_A = 0 \qquad J_B = 3$$

$$\underline{x} \varepsilon \psi_B \qquad J_A = 3 \qquad J_B = 0$$

$$\underline{x} \varepsilon \psi_{AB} \qquad J_A = 1 \qquad J_B = 1$$

$$\underline{x} \varepsilon \psi_D \qquad J_A = 2 \qquad J_B = 2$$

In this case players A and B attempt to <u>minimize</u> J_A and J_B, respectively. Notice that if there is a "duel to the death" (eliminating the possibility of outcome 4) or if B and A order outcomes 3 and 4 differently, or if they consider them equivalent, then a zero-sum game may be formulated.

It is theoretically possible to solve a game of the type just described. In practice, however, the solution would be most difficult to obtain except, perhaps, in the simplest of cases. Moreover, the assignment of numerical payoffs is clearly arbitrary. What is of prime importance here is the determination of which outcome will occur; indeed, the dogfight problem is really a complicated "game of kind." This fact, along with other considerations to be discussed below, suggests an alternative approach to the dogfight problem.

In a pursuit-evasion game the roles of the players are fixed throughout the play of the game: one player is a pursuer, the other an evader. This is not necessarily the case in a dogfight. In fact, one distinguishing feature of the dogfight is that the players may change roles several times during the course of the engagement. Consequently, one can think of the dogfight problem as being comprised of the following two parts:

a. Given certain information regarding the states of the two vehicles, determine the roles to be assumed by each vehicle. Does A chase B or vice-versa?

b. Given the roles of the two vehicles, determine the associated optimal strategies or guidance laws.

The second problem is just the pursuit-evasion problem. The first problem, which we call the "role selection" problem, is the unique and crucial aspect of the dogfight problem; we will be concerned primarily with this problem in the remainder of this chapter.

Conceptually, the solution to the two interrelated problems may be visualized as follows. The basic problem is characterized by the termination criteria embodied in (1)-(4) above. In principle, once the initial state of the problem is given, the outcome is also known, assuming optimal play by both sides. Thus, one can consider the state space being divided (in principle) into four distinct regions: W_A (win for A), W_B (win for B), W_{AB} (mutual destruction), and W_D (draw). If play starts in W_A and A plays optimally, then the state remains in W_A for the duration of the game (and terminates in ψ_A which is, of course, contained in W_A). A similar statement may be made with respect to W_B. In the

other two regions, W_{AB} and W_D, both players must play optimally in order for the state to remain in the given region. If one player does not use his optimal strategy in either of these regions, the state can move to a region that is more favorable to his opponent. The description here is reminiscent of the situation in the pursuit-evasion game of kind. In fact, the boundaries between the various regions ($W_A,...,W_D$) constitute "barriers" in the usual Isaacs' sense (Ref.3), i.e., they are not crossed in optimal play.

The method for selecting a player's role is now obvious, at least in regions W_A and W_B. If $\underline{x} \epsilon W_A$, A is the pursuer, B the evader and vice versa if $\underline{x} \epsilon W_B$. Moreover, for play in one of these regions a payoff such as time to capture may be superimposed on the game so that unique optimal strategies are determined. (This makes the problem a game of degree in that region.)

The situation in the remaining two regions is not so clear. In effect, both players are pursuers in W_{AB} and both are evaders in W_D. However, the prime goal of each player is to avoid being driven into a less favorable region and each one hopes his opponent will make a mistake he can take advantage of. There does not appear to be any rationale for superimposing a payoff on either of these regions.

An Example

To give a clearer picture of the ideas just discussed, we shall consider a very simple example. Many complicated features are neglected in order that some direct results can be developed with only intuitive arguments.

Problem Formulation--Consider two vehicles, A and B, that
move in the horizontal plane. Both vehicles have exactly the
same performance characteristics: the same fixed speed V and the
same minimum allowable turning radius R. Both navigate by select-
ing an appropriate radius of turn at each time instant. For sim-
plicity it is assumed that the effective firing envelope of each
craft is merely a line segment of length ℓ. It is fixed relative
to the craft and is oriented along the direction of the velocity
vector. This situation is represented in Figure 9.

The game is such that if A ever intersects the line segment
ℓ_B, A loses; and if B ever falls upon line segment ℓ_A, B loses.
Therefore, the goal of each player is to force the antagonist into
his firing envelope while keeping himself out of the firing en-
velope of his opponent. It is assumed that fatality is instant
so that the game terminates whenever one aircraft enters the
other's firing envelope.

Only the relative kinematic equations need be considered here.
Point A may be taken as the origin with the velocity vector V_A
coincident with the y-axis. Then x and y specify the position of
B relative to A and θ specifies the relative flying direction of
B with respect to A. (See Figure 9)

Consequently, our game evolves in a three dimensional (rela-
tive) state space with $-\infty < x,y < +\infty$ and $0° \leq \theta \leq 360°$. At
time t, the entire information as to the state of the two aircraft
is specified as a point in this state space. Under the (relative)
combined maneuvers of A and B, this state point $\chi = (x,y,\theta)$ evolves
from some given initial state $\chi_0 = (x_0,y_0,\theta_0)$ at t=0 to some ter-
minal state $\chi_f = (x_f,y_f,\theta_f)$. The terminal state is defined as
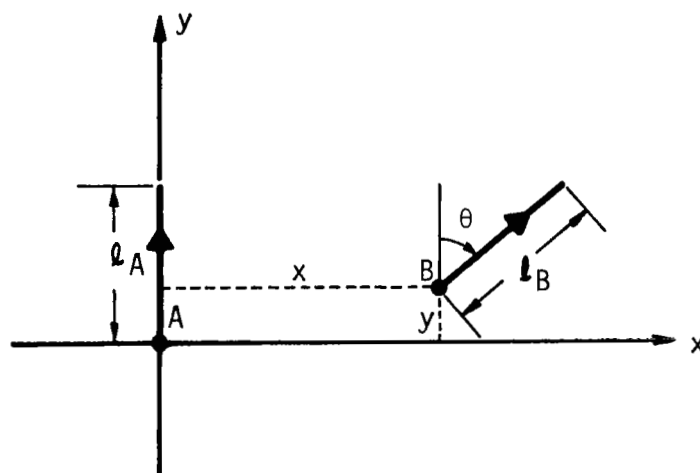
$R_A = R_B = R =$ MINIMUM RADIUS OF TURN

$\ell_A = \ell_B = \ell =$ "EFFECTIVE" FIRING RANGE

$v_A = v_B = v =$ AIRCRAFT SPEED

FIG. 9 SIMPLIFIED REPRESENTATION OF AIR-TO-AIR
COMBAT OF IDENTICAL VEHICLES. (MOVING
COORDINATE SYSTEM CENTERED AT A.)

the state where at least one of the aircraft is inside the fire range of the other and the combat game terminates (ignoring, for now, the condition corresponding to the draw).

To consider the terminal state in this relative space, refer again to Figure 9. Since point A is always fixed, it is clear that whenever χ is such that x = 0, $0 \leq y \leq \ell$, aircraft B will be destroyed. In other words, define

$$\psi_A = \{(x,y,\theta) \mid x=0,\ 0 \leq y \leq \ell,\ \text{for any } \theta\}$$

Thus, ψ_A consists of all those terminal states favorable to A, i.e., if $\chi_f \epsilon \psi_A$ then B loses. (See Fig.10)

The structure of those terminal states favorable to craft B, ψ_B, is a bit more complicated. Consideration of several special cases in Figure 11 will make it clear. When $\theta_f = 0$, the favorable states are defined by B being directly behind A at a distance less than ℓ (Fig. 11(a)). When $\theta_f = 90°$, the states are such that B is just at A's left side with distance less than ℓ, (Fig.11 (b)). Similarly, the cases $\theta_f = 180°$, 270° are also clear from Figure 11(c) and (d), respectively.

Returning to Figure 10 it is now easy to understand that ψ_B is a helical type of surface. This surface is bounded by the θ-axis and by a helical curve with horizontal generatrix of length ℓ.

The union of ψ_A and ψ_B, $\psi_A \bigcup \psi_B$, comprises all of the possible terminal states for which at least one of the aircraft is knocked down and the combat game ends. It should be noted from Figure 10
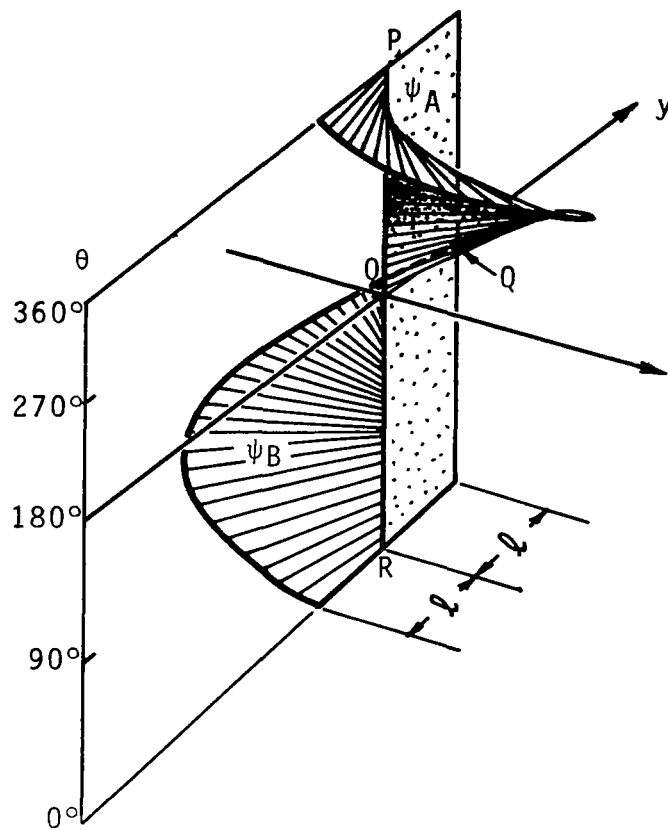
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FIG.10 TERMINATION SURFACES:

ψ_A = TERMINAL STATES FAVORABLE TO A

ψ_B = TERMINAL STATES FAVORABLE TO B

(a) $\theta_f = 0°$ (b) $\theta_f = 90°$ (c) $\theta_f = 180°$ (d) $\theta_f = 270°$
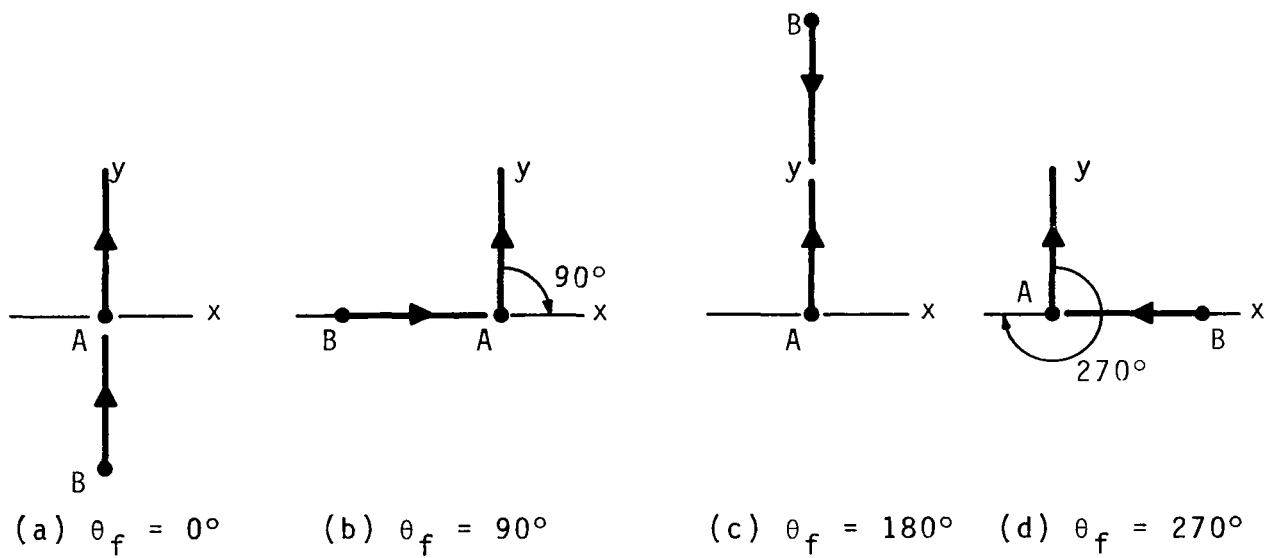
FIG.11 TERMINAL ORIENTATIONS FAVORABLE TO B

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that there are some states common to both ψ_A and ψ_B, namely the line segments POR and OQ which constitute the intersection set ψ_{AB}. On ψ_{AB} both A and B will be destroyed — by either collision or mutual firing.

Our problem is: given any initial state $\chi_o = (x_o, y_o, \theta_o)$ in the x-y-θ space, what is the outcome of the game when A and B play optimally with respect to their individual goals? We discuss this question below.

<u>Solution Characteristics</u>--In this section we examine some of the more interesting characteristics of the solution to the problem just posed. In particular, we illustrate the nature of the various regions of the state space that correspond to the possible outcomes of the dogfight. We also present some specific optimal trajectories. We do not, however, give the <u>complete</u> solution to the problem.

Let us assume, for the moment, that A and B are engaged in a duel to the death. Thus, we eliminate, for now, the subspace W_D corresponding to both vehicles escaping. Since ψ_A and ψ_B are surfaces, and ψ_{AB} is reduced to a curve that is the boundary between ψ_A and ψ_B, we can argue that W_{AB}, which connects with ψ_{AB}, serves to separate W_A and W_B. The state space decomposition might then look something like that shown in Figure 12.

Whenever the initial state is to the left side of W_{AB}, the game results will be favorable to A; whenever the initial state is to the right side of W_{AB}, the game results will be favorable to B. If initially the state is in W_{AB}, then it should be maintained in it, and the outcome would be a mutual destruction, i.e., the trajectory runs somehow within space W_{AB} until it hits ψ_{AB}.
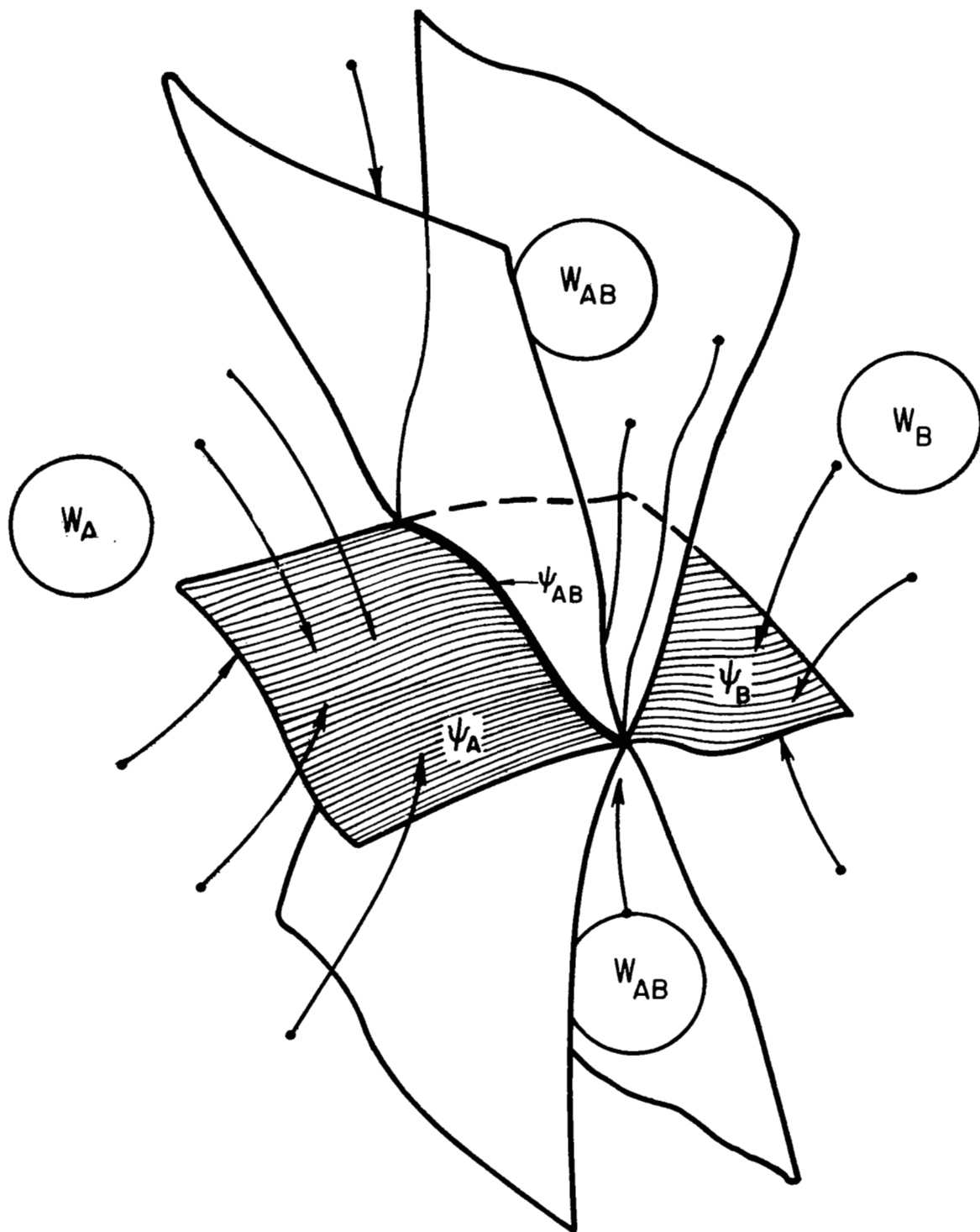
FIG.12   QUALITIVE ILLUSTRATION OF STATE-SPACE DECOMPOSITION

Now we return to Figure 10 and try to define the subspaces W_A, W_B and W_{AB}. Inasmuch as the vehicles have equal capabilities, one would expect the terminal surfaces ψ_A and ψ_B to be essentially the same. This not being the case in the given coordinate system, a coordinate transformation 's suggested. Consider Figure 13: we keep the origin A fixed but rotate the x-y frame with angle $\bar{\theta}$, where $\bar{\theta}$ is half the angle between the vector V_A and V_B. The new state θ ranges from -90° to +90°, and ψ_A and ψ_B in the new coordinates are as shown in Figure 14. The structure is now two helical surfaces that are symmetric with respect to both the origin and the plane y=0. It is then obvious that the decomposition of the space should also be symmetric with respect to the origin and the plane y=0. Since W_{AB} has to pass the line segments PQ,RS,POR, the plane y=0 has to belong to W_{AB} ($\{y=0\} \subset W_{AB}$). It follows that $W_A \subset \{y>0\}$ and $W_B \subset \{y<0\}$. Insofar as aircraft A and aircraft B have exactly the same performance characteristics, it is natural that they have exactly the same chances to lose or to win. So the volumes of W_A and W_B should be equal and their shapes should be symmetric. †

To understand more about W_{AB}, let us consider the trajectories on the plane y=0, or the case that aircraft B is always on the x-axis (see Fig.15a). The two craft are turning symmetrically and simultaneously until they are opposed to each other then they approach each other and open fire when the distance is short enough. The relation between x and θ is $x = \text{constant} \pm 2R\cos\theta$. The optimal trajectories are then those shown in Figure 15b. They will stay on the plane y=0, and the whole plane y=0 belongs to W_{AB}. No one-side-win or draw can exist there.

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†On the other hand, if aircraft A is more capable than B (for example, A has greater speed, smaller turning radius, or longer fire range, etc.), A should have better chance to win. In this case, W_A will be larger than W_B and W_{AB} will be bent away from W_A.
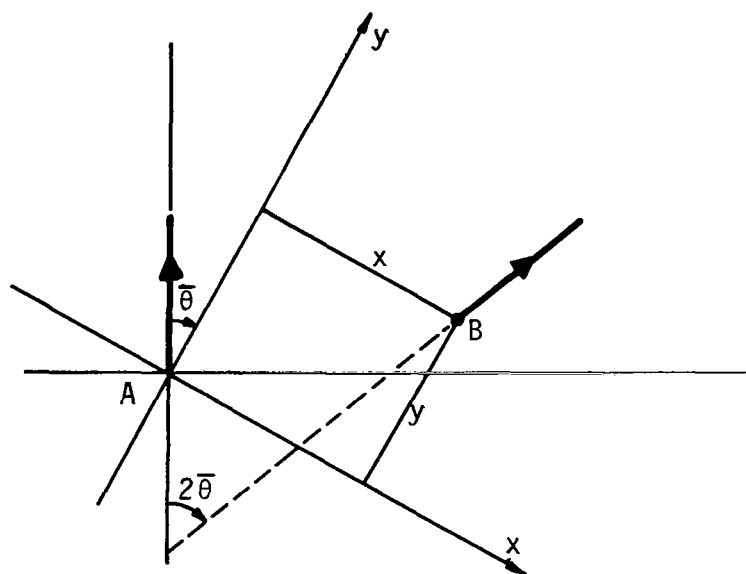
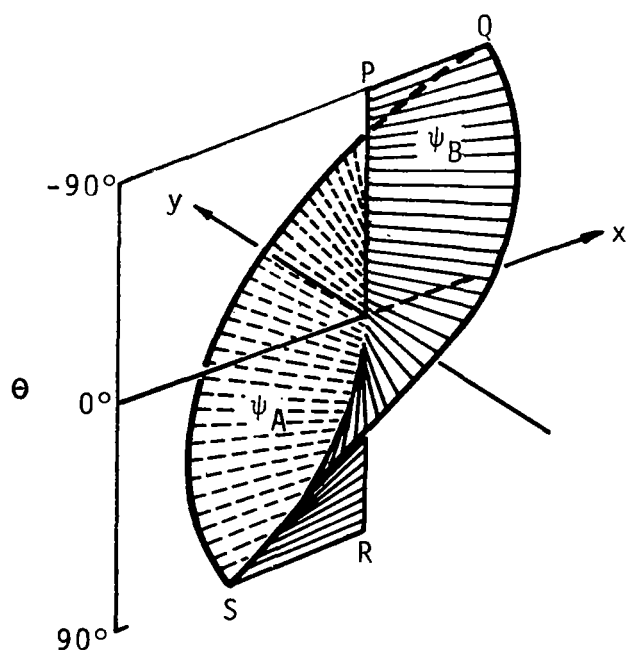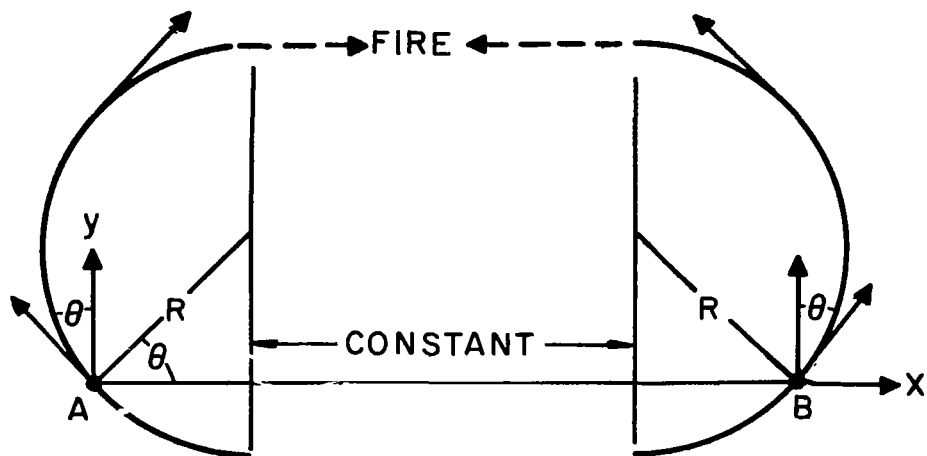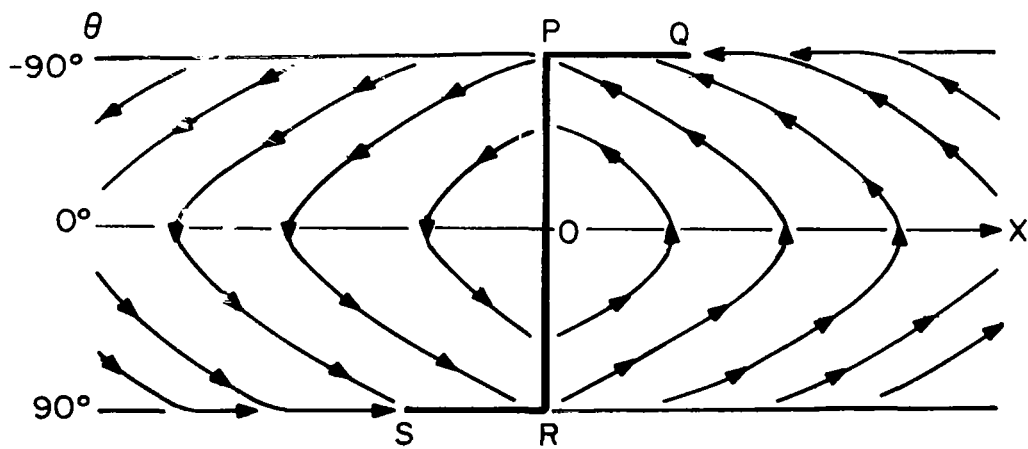FIG.13   COORDINATE ROTATION OF THE RELATIVE STATE SPACE

FIG.14   TERMINATION SURFACES IN NEW COORDINATE SYSTEM

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a) PATH GEOMETRY

b) OPTIMAL TRAJECTORIES

FIG.15 PATHS FOR MUTUAL DESTRUCTION IN THE PLANE y = 0

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Another special case of interest is the plane $\theta=90°$ (or $\theta=-90°$). Here the velocity directions of A and B are opposed to each other with a difference of 180° (Fig. 16). Then, A and B can always exterminate each other, if both make an outside turn (case (a)) and then open fire when they meet. In this case the whole $\theta=90°$ plane would belong to W_{A^-}; W_D does not exist here. However, if we assume, *at this time*, that the two players prefer having a draw to mutual extermination, their optimal trajectories should be an inside-turn, like that in Figure 16(b). If it is impossible to do so, they can just fly straightly and forget each other. Anyway, they can always avoid total extermination. The whole plane $\theta=90°$, except the line segment $(-\ell \leq x \leq 0, y=0)$, would then belong to W_D. *The decomposition of the space depends very much on our assumptions as to pilot preference.*

The subspace of a draw, W_D, is defined as the union of those states that do not result in termination on ψ_A, ψ_B or ψ_{AB}. Each pilot can not change a trajectory in W_D unilaterally to his advantage. So, trajectories starting in W_D should always stay in W_D without crossing the boundaries to either W_A, or W_B, or W_{AB}. Two special types of trajectories in W_D may be distinguished: (1) stationary points, that is points for which the two craft never change their relative position in subsequent maneuvering, (2) closed loops, i.e., the initial state point moves, but it follows some cycle periodically (neither of the players can break the stalemate himself). Other irregular trajectories that are not stationary points or closed loops, but always remain in W_D are also possible. Undoubtedly, the shape of W_D will be very complicated. To describe it completely is beyond the scope of this report. Here, we are just going to explore some special sections of W_D to show its relationship to the other regions. The main purpose is to get a feeling for the ideas previously presented.
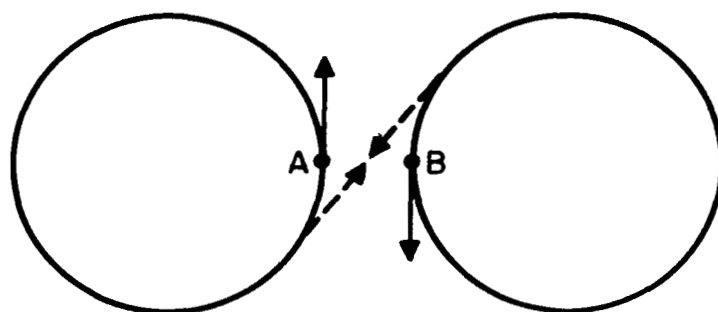
a) MUTUAL-DESTRUCTION PATH
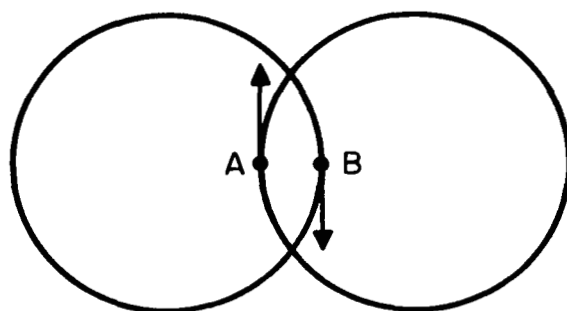
b) PATH FOR A DRAW

FIG.16   POSSIBLE PATHS IN PLANE Θ=90°

Consider the situation in which one craft is directly behind the other but the relative distance between them is greater than the firing range, ℓ. In such a case the lead vehicle must continue to dash (i.e., fly a straight course at max-speed) since any turning maneuver would shorten the distance between the him and his "pursuer." Since both vehicles have the same capabilities, the relative positions remain fixed and we have a "stationary" draw. States for which this is the case lie on the two half-lines $M\text{-}y_\infty$ and $N\text{-}y_{-\infty}$ shown in Figure 17.[†] A second kind of "stationary" draw involves turning maneuvers. In particular, it occurs when both vehicles are turning in the same circle but their relative position remains fixed. For such to be the case their position must be as shown in Figure 18, i.e., the angle AOB must be equal to 2θ and B must be in the $y\text{-}\theta$ plane. The locus of these points is given by $x=0$, $y = \pm 2R \sin \theta$, where R is the minimum turning radius. Curves EOF and GOH in Figure 17 correspond to initial states for this kind of stationary draw.

To provide an understanding of how the "barriers" arise and separate the regions, consider the case shown in Figure 19. The position of B is to the left and rear of A and the value of θ is negative. Thus, the initial state corresponds to points like U, J or V in Figure 17. Now B has better tactical position than A has. The tactic for A is to make a full right turn to evade, whereas that for B is to make a full right turn to pursue.[††] After following some arcs of their respective circles, they fly straightly along the common tangent line of the two circles (see Fig. 19).

[†]Notice that if mutual destruction is preferred to a draw, not all of these lines will be in W_D — part of them will belong to W_{AB}.

[††]In an actual combat situation A might turn into B inasmuch as he is not likely to be "killed" by an instantaneous traverse of the firing cone.
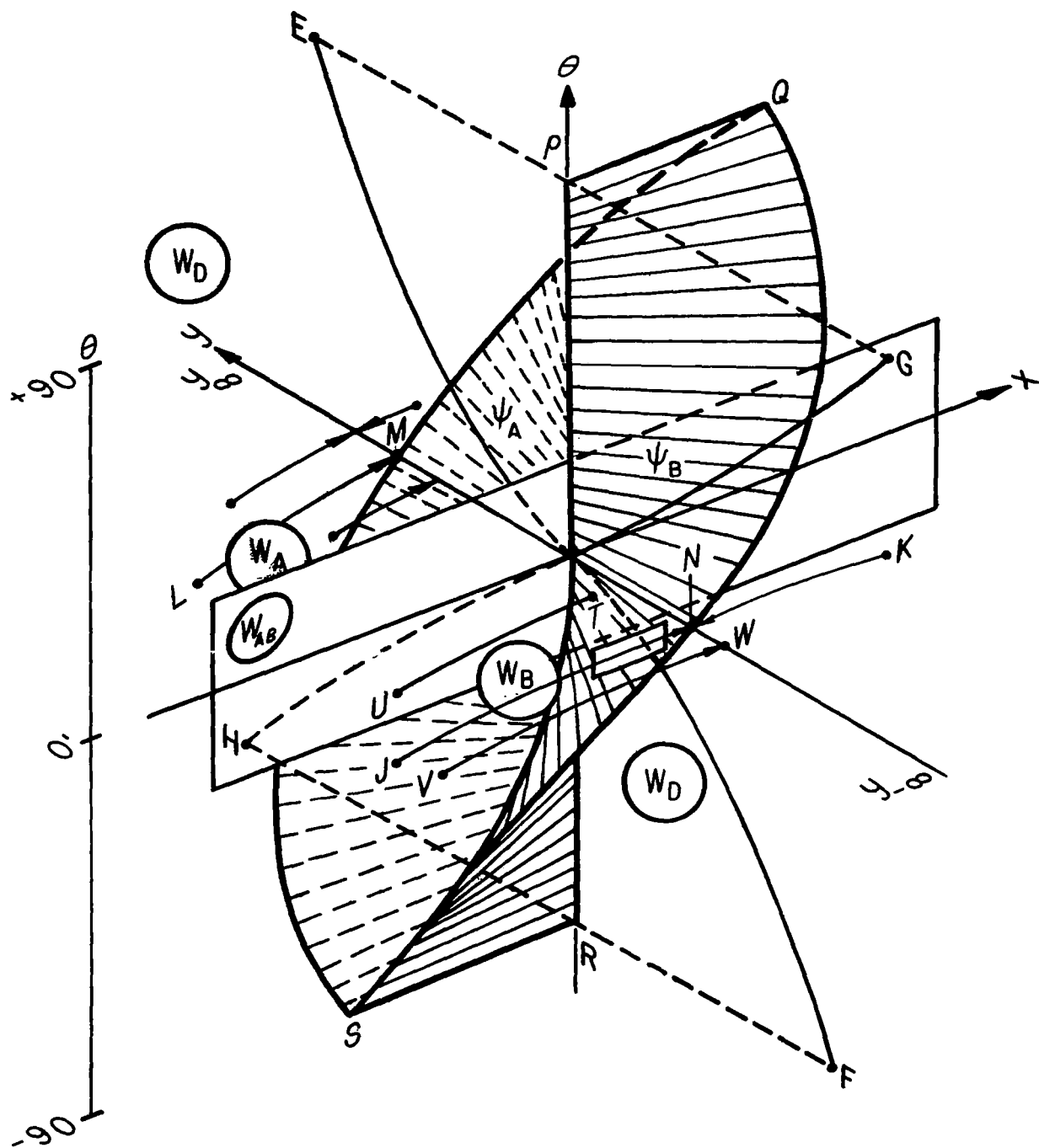
FIG.17   PARTIAL STATE-SPACE DECOMPOSITION FOR SIMPLE EXAMPLE
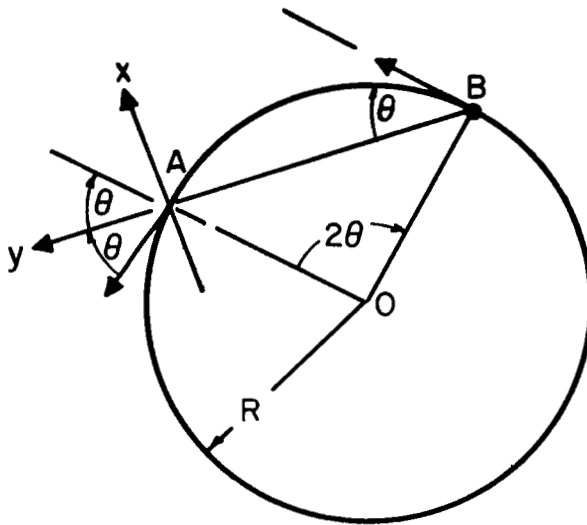
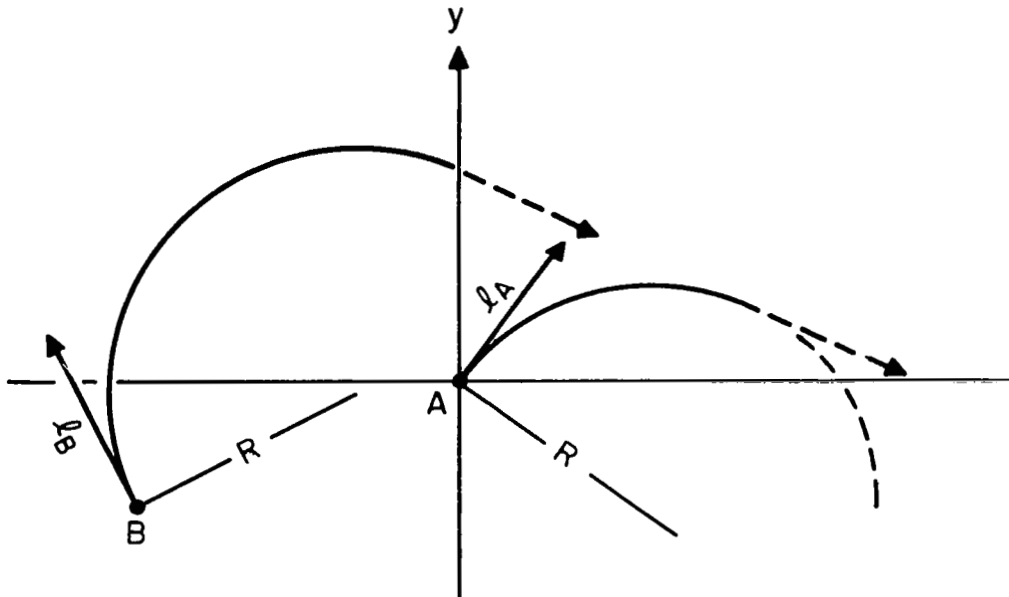FIG.18   GEOMETRY FOR "STATIONARY" DRAW INVOLVING TURNING MANEUVERS

FIG.19   GEOMETRY ILLUSTRATING THE FORMATION OF "BARRIERS"

If originally, B is so far away from A that, after turning, B's fire range is still too far back from A, the two vehicles just keep flying, and the final result is a draw. In Figure 17, this corresponds to the trajectory VW. We understand that $VW \epsilon W_D$.

If initially, B is near enough to A so that B can shoot A down during the turn or after finishing the turn the corresponding trajectory will look like curve UT in Figure 17. T is a point on the surface ψ_B, where aircraft A is destroyed. So $UT \epsilon W_B$.

However, there must be some cases when the initial distance between A and B is such that, after turning, the front tip of B's fire range, ℓ_B, is just at the margin of reaching aircraft A. Any increase of their initial distance would make B miss A, while, any decrease of this distance would guarantee B destroying A. This marginal case corresponds to the trajectory arc of JN in Figure 17. Hence, N is just on the rim of ψ_B.

Some of the other marginal optimal trajectories corresponding to the other initial states are drawn as arcs LM and KN in Figure 17. We can now assert that arcs JN and KN must be on the boundary surface that divides W_D and W_B, while arc LM must be on the boundary that divides W_D and W_A. Thus, these curves constitute portions of the appropriate barriers. To illustrate, the situation for the plane $\theta=0°$ is shown qualitatively in Figure 20.

Summary

In this chapter we considered the dogfight problem. We saw that this problem was comprised of two parts: namely, "role-selection" and pursuit-evasion. We developed an approach to the role-selection problem that involves decomposing the state space
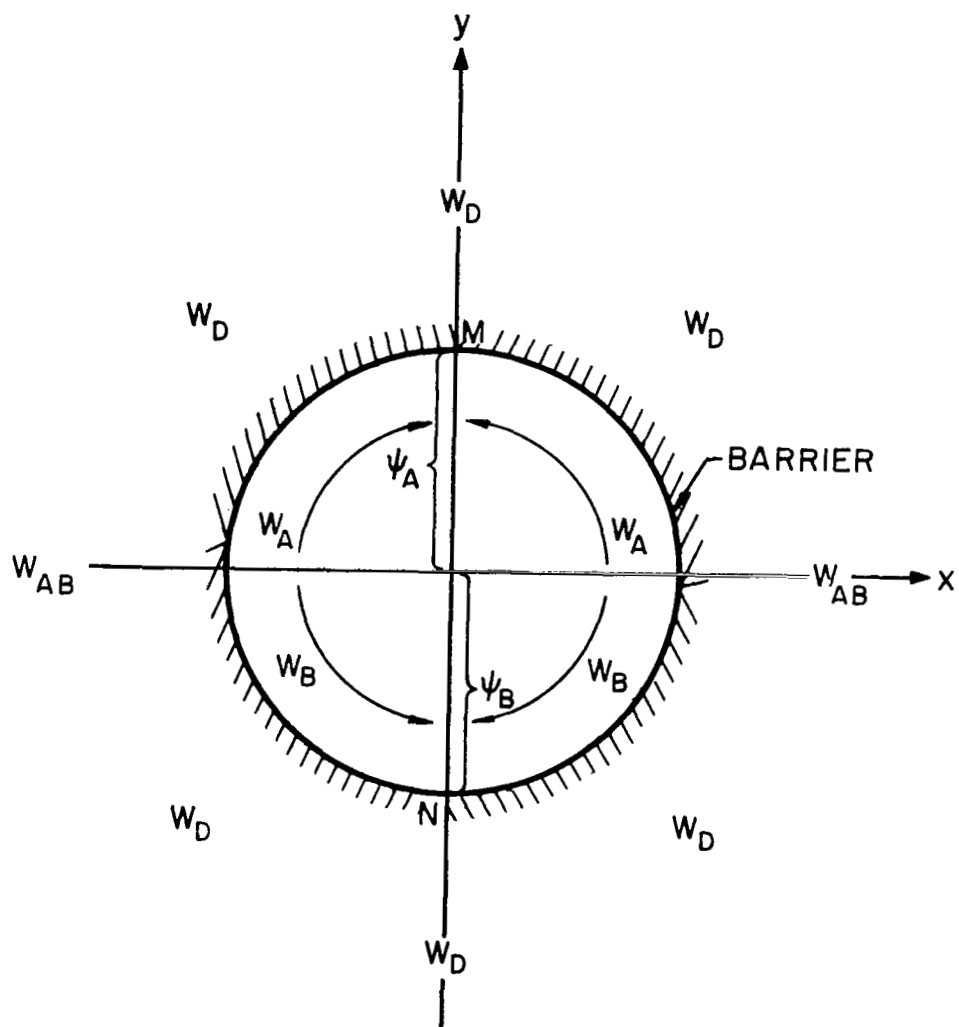
FIG.20   QUALITIVE SKETCH OF STATE-SPACE
         DECOMPOSITION IN PLANE Θ = 0°

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into regions that correspond to the (four) possible outcomes of an aerial duel. This approach appears to be innovative with respect to the dogfight problem, although it is closely related to Isaacs' analytic and geometric methods for pursuit-evasion "games of kind."

A simple dogfight problem was analyzed in some detail in order to illustrate the ideas involved in our approach. The difficulties encountered in this analysis suggest that solving realistic dogfight problems in this manner may well be impractical. However, a combination of the concepts introduced in this chapter and the computational techniques developed earlier may prove fruitful.

6. CONCLUDING REMARKS

In this report we have presented the salient results of a study to apply differential game theory to aerial combat problems. In particular, we examined the role of differential game theory in aerial combat problems (Chapter 2); we developed a new approach to manned aerial combat games (Chapter 3) and applied it to two classical differential game problems (Chapter 4); lastly, we suggested an approach to the "dogfight" problem and illustrated the approach by a simple example (Chapter 5).

The examination of the state-of-the art of differential game theory revealed the need for computational approaches to solving the "games" problems associated with aerial combat. We developed such an approach by reformulating the problem. The new formulation was arrived at from a consideration of the information upon which pilots base control decisions and of the nature of the decisions themselves.

As we view it, the aerial combat game is played in a discretized state-space. The discrete states correspond to "blocks" in the continuous state-space; these blocks reflect the inability of a pilot to "locate" his opponent precisely. A pilot's control decision is a choice of one out of a finite set of "canonical control maneuvers." Thus, knowing that his opponent is located within a given block, each pilot must decide which of his available maneuvers would best serve his own goal.

The above assumptions were shown to reduce the aerial combat game to a discrete Markov game. Computational methods for solving such games were available in the literature. These methods were

modified using concepts of state increment dynamic programming.
The modifications are appealing from a physical standpoint, and
they result in a tremendous saving of computation time and high-
speed memory requirements.

The computation scheme was used to solve analogues of two
classic differential game problems, the "Homicidal Chauffeur
Problem," and the "Two-Car Problem." These applications demon-
strate the basic validity of the approach and also gave an indica-
tion of its future usefulness.

Our approach to the dogfight problem embodied concepts from
both nonzero-sum games and games of kind. Essentially, the dog-
fight problem was considered as having two parts: namely, the
problem of deciding which player should be a pursuer and which an
evader (role-selection problem), and, the pursuit-evasion problem.
We concentrated on the role-selection problem and showed how the
state-space might be decomposed into regions corresponding to the
four possible outcomes of a dogfight. These regions, which are
separated by "barriers," can serve to define the roles of the
players. A simple example was studied to illustrate the ideas
and difficulties involved in this approach to the dogfight problem.

Problem Areas and Suggestions for Further Research

The conceptual-computational approach we have taken to aerial
combat problems seems to have great potential. As noted earlier,
stochastic effects are absorbed in the transition probability
matrix and are thus accounted for directly and simply. Convergent
algorithms are available for solving the resulting Markov game
and the solutions are of the feedback type. The physically mean-
ingful aspects of the aerial combat problem are thus directly

revealed. One need not be concerned with determining the various singular surfaces that characterize differential game solutions as described by Isaacs. Nonetheless, our approach is in its infancy and there are problems that need to be resolved. We discuss some of these below. For discussion purposes, it is convenient to divide the problems into two categories: computational and analytical (or theoretical). However, one must realize that computational and analytical aspects of the method are closely intertwined.

Computational Problems--The most pressing problem with our approach in its present form concerns the computational load and storage requirements associated with the method. While the computational demands are not as excessive as those characteristic of dynamic programming's "curse of dimensionality," they are nevertheless considerable. The requirement for obtaining the transition probabilities $p_{ij}(u,v)$, for all i,u, and v, imposes the largest computational burden. While each computation is simple in itself, a realistic problem could necessitate over one million such calculations. A related problem is the storage of the p_{ij}'s. Rapid access core storage may not be possible and reading and writing from other storage devices (tapes, discs, etc.) may be necessary. This would increase the effective computation time.

It should be noted that the p_{ij}'s can be computed independently of the optimization algorithm, i.e., before solving the optimization problem. Inasmuch as the p_{ij}'s are independent of the payoff, it would be possible to change payoffs without having to recompute the transition probabilities. In addition, insofar as it is possible to account for constraints by "penalties" in the payoff, studying the effects of changing these constraints

would not require recomputing the p_{ij}'s. On the other hand, if changes in vehicle characteristics are to be studied and if these changes can only be accounted for by altering the equations of motion, then a new set of transition probabilities would have to be computed for each change. This implies that certain parametric studies could be quite expensive computationally.

Another computational phase of the approach is the iterative solution of the discrete Markov game. This aspect of the computation is less demanding than is the calculation of the transition probabilities but it is, nonetheless, nontrivial. It appears that approximately one-third of the total computational effort in obtaining a solution will be consumed in the iterative procedure.

The above computational requirements raise questions as to the feasibility of solving realistic problems using this approach. It is natural to ask, for example, how large a problem could be solved at a reasonable cost.[†] Answering this question will require formulating specific problems and estimating the associated computational requirements and costs. This appears to be a most appropriate avenue to pursue. In fact, it seems manifestly desirable to apply the technique, as currently formulated, to what might be termed a realistic problem. Such an application would provide answers to a meaningful problem as well as an indication of the computational feasibility of the approach.

[†]In making such a determination one must, of course, weight the value of a particular solution against the cost for obtaining it. In doing this, it must be kept in mind that a feedback (or a global) solution is obtained using our approach and that such a solution is of much greater value than an open-loop answer for one set of initial conditions.

Methods for reducing the computational requirements should also be investigated. An important area for research is the examination of techniques for ameliorating the load associated with computing the p_{ij}'s. One obvious technique would be to use a coarser, nonuniform, state-space discretization. Another is to neglect (or prohibit) transitions across the "edges" or "corners" of a state "block." Programming techniques, such as word-packing schemes, that help alleviate storage requirements should also be investigated. Finally, more rapid algorithms for iterating on the game solution should be sought. Possibilities range from Howard's "iteration in policy space" (Ref.33) to using nonlinear programming and "over-relaxation" methods.

Analytical Problems--There are a number of questions of an analytical nature concerning the approach that also present themselves. We mention just a few of them here.

One of the more important analytical questions concerns the existence of a saddle-point solution to the Markov game. Currently, we are assured of obtaining a min-max (or max-min) solution to the problem we have posed (i.e., solutions to the so-called majorant games). We could compare max-min and min-max answers; if they are equal, we have a saddle-point. However, this requires two iterative solutions, and a better alternative is to seek necessary and sufficient conditions for a saddle-point to exist. In this connection we mention that the satisfaction of Eq. (3.13) guarantees the existence of a saddle-point. The condition is analogous to requiring the Hamiltonian to have a saddle-point. Unfortunately, the condition can only be verified if analytic expressions for the p_{ij}'s are available and such is not likely to be the case. It will be noted that if the p_{ij}'s and the c_{ij}'s are separable with respect to u and v, then the satisfaction of Eq. (3.13) is obvious. In many physical problems one might expect this separation to exist.

We consider the Markov game to be the fundamental problem and not an approximation to a continuous differential game. Nevertheless, it would be useful to determine the relation of the Markov game to a corresponding continuous differential game. In particular, one might like to know under what conditions the solution to the Markov game approaches that of the corresponding differential game. To answer this question the limiting behavior of the solution to the Markov game would have to be investigated as the discretization becomes finer.

In the two applications of our approach, the problems were formulated with both players having the same information sets. The method should be extended to the case where the information sets for the two players are different. This will be especially important if the approach is to be used for the "dogfight" problem. The manner in which this extension is to be accomplished is an area for study. Specifying the "canonical maneuvers" for the two examples was a trivial matter. Determining appropriate canonical maneuvers for more realistic problems will not be so easy and will require further study.

Dogfight Problem--As noted earlier, the analytic-geometric approach to the dogfight problem outlined in Chapter 5 is not likely to provide numerical answers for reasonably realistic problems. This is not surprising in view of the relation of this approach to the more "classical" methods of attacking differential games. However, the approach is useful conceptually and it might be beneficial to attempt to employ it in conjunction with the computational technique we have developed. The manner in which this could be done is not obvious and is an area for study.

Alternatively, we might attempt a direct formulation of the dogfight problem as a Markov game. Just how this formulation should proceed is not immediately apparent. Nor can we be sure that the approach is a viable one. These are problems that require further investigation.

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